IX. *On the Tidal Friction of a Planet attended by several Satellites, and on the Evolution of the Solar System.*

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Introduction.

In previous papers on the subject of tidal friction* I have confined my attention principally to the case of a planet attended by a single satellite. But in order to make the investigation applicable to the history of the earth and moon it was necessary to take notice of the perturbation of the sun. In consequence of the largeness of the sun's mass it was not there requisite to make a complete investigation of the theory of a planet attended by a pair of satellites.

In the first part of this paper the theory of the tidal friction of a central body attended by any number of satellites is considered.

In the second part I discuss the degree of importance to be attached to tidal friction as an element in the evolution of the solar system and of the several planetary sub-systems.

* Phil. Trans, Parts I. and II., 1879, and Part II. 1880; Proc. Roy. Soc., No. 197, 1879, and No. 202, 1880.

The last paragraph contains a discussion of the evidence adduced in this part of the paper, and a short recapitulation of the observed facts in the solar system which bear on the subject. This is probably the only portion which will have any interest for others than mathematicians.

I.

THE THEORY OF THE TIDAL FRICTION OF A PLANET ATTENDED BY ANY NUMBER OF SATELLITES.

§ 1. *Statement and limitation of the problem.*

Suppose there be a planet attended by any number of satellites, all moving in circular orbits, the planes of which coincide with the equator of the planet; and suppose that the satellites all raise tides in the planet. Then the problem proposed for solution is to investigate the gradual changes in the configuration of the system under the influence of tidal friction.

This problem is only here treated under certain restrictions as to the nature of the tidal friction and in other respects. These limitations however will afford sufficient insight into the more general problem. The planet is supposed to be a homogeneous spheroid formed of viscous fluid, and the only case considered in detail is that where the viscosity is small; moreover, in the tidal theory adopted the effects of inertia are neglected. I have however shown elsewhere that this neglect is not such as to materially vitiate the theory.* The satellites are treated as attractive particles which have the power of attracting and being attracted by the planet, but have no influence upon one another. A consequence of this is that each satellite only raises a single tide in the planet, and that it is not necessary to take into consideration the actual distribution of the satellites at any instant of time. We are thus only concerned in determining the changes in the distances of the satellites and in the rotation of the planet.

If the mutual perturbation of the satellites were taken into account the problem would become one of the extremest complication. We should have all the difficulties of the planetary theory in determining the various inequalities, and, besides this, it would be then necessary to investigate an indefinitely long series of tidal disturbances induced by these inequalities of motion, and afterwards to find the secular disturbances due to the friction of these tides.

It is however tolerably certain that in general these inequality-tides will exercise a very small influence compared with that of the primary tide. Supposing a relationship between the mean motions of two, three, or more satellites, like that which holds good in the Jovian system, to exist at any epoch, then it is not credible but that such relationship should be broken down in time by tidal friction. General considerations

* "Problems Connected with the Tides of a Viscous Spheroid—No. III.," Phil. Trans., Part. II., 1879.

would lead one to believe that the first effect of tidal friction would be to set up amongst the satellites in question an oscillation of mean motions about the average values which satisfy the supposed definite relationship; afterwards this oscillation would go on increasing indefinitely until a critical state was reached in which the average mean motions would break loose from the relationship, and the oscillation would subsequently die away. It seems probable therefore that in the history of such a system there would be a series of periods during which the mutual perturbations of the satellites would exercise a considerable but temporary effect, but that on the whole the system would change nearly as though the satellites exercised no mutually perturbing power.

There is however one case in which mutual perturbation would probably exercise a lasting effect on the system. Suppose that in the course of the changes two satellites came to have nearly the same mean distance, then these two bodies might either come ultimately into collision or might coalesce so as to form a double system like that of the earth and moon, which revolve round the sun in the same period. In this paper I do not make any attempt to trace such a case, and it is supposed that any satellite may pass freely through a configuration in which its distance is equal to that of any other satellite.

§ 2. *Formation and transformation of the differential equations.*

In this paper I shall have occasion to make frequent use of the idea of moment of momentum. This phrase is so cumbrous that I shall abridge it and speak generally of angular momentum, and in particular of rotational momentum and orbital momentum when meaning moment of momentum of a planet's rotation and moment of momentum of the orbital motion of a satellite. I shall also refer to the principle of conservation of moment of momentum as that of conservation of momentum.

The notation here adopted is almost identical with that of previous papers on the case of the single satellite and planet; it is as follows:—

For the planet let:

M=mass; a=mean radius; g=mean pure gravity; w=mass per unit volume; υ=viscosity; $\mathfrak{g}=\frac{2}{5}g/a$; n=angular velocity of rotation; C=moment of inertia about the axis of rotation, and therefore, neglecting the ellipticity of figure, equal to $\frac{2}{5}Ma^2$.

For any particular one of the system of satellites, let:

m=mass; c=distance from planet's centre; Ω=orbital angular velocity.

Also μ being the attraction between unit masses at unit distance, let $\tau=\frac{3}{2}\mu m/c^3$; and let $\nu=M/m$.

These same symbols will be used with suffixes 1, 2, 3, &c., when it is desired to refer to the 1st, 2nd, 3rd, &c., satellite, but when (as will be usually the case) it is desired simply to refer to any satellite, no suffixes will be used.

Where it is necessary to express a summation of similar terms, each corresponding to one satellite, the symbol Σ will be used; *e.g.*, $\Sigma\kappa c^{\frac{1}{2}}$ will mean $\kappa_1 c_1^{\frac{1}{2}}+\kappa_2 c_2^{\frac{1}{2}}+$ &c.

Now consider the single satellite m, c, Ω, &c.

If this satellite alone were to raise a tide in the planet, the planet would be distorted into an ellipsoid with three unequal axes, and in consequence of the postulated internal friction, the major axis of the equatorial section of the planet would be directed to a point somewhat in advance of the satellite in its orbit.

Let $\mathfrak{f}$ be the angle made by this major axis with the satellite's radius vector; $\mathfrak{f}$ is then a symbol subject to suffixes 1, 2, 3, &c., because it will be different for each satellite of the system.

Then it is proved in (22) of my paper on the "Precession of a Viscous Spheroid,"* that the tidal frictional couple due to this satellite's attraction is $C\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathfrak{f}$.

Now it appears from Sec. 14 of the same paper that the tidal reaction, which affects the motion of each satellite, is independent of the tides raised by all the other satellites.

Hence the principle of conservation of momentum enables us to state, that the rate of increase of the orbital momentum of any satellite is equal to the rate of the loss of rotational momentum of the planet which is caused by that satellite alone. The rate of loss of this latter momentum is of course equal to the above tidal frictional couple.

When the planet is reduced to rest the orbital momentum of the satellite in the circular orbit is $\Omega c^2 Mm/(M+m)$. Hence the equation of tidal reaction, which gives the rate of change in the satellite's distance, is

$$\frac{d}{dt}\left[\frac{Mm}{M+m}\Omega c^2\right]=C\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathfrak{f}\,. \qquad (1)$$

A similar equation will hold true for each satellite of the system.

This equation will now be transformed.

By KEPLER'S law $\Omega^2c^3=\mu(M+m)$ and therefore

$$\frac{Mm}{M+m}\Omega c^2=\mu^{\frac{1}{2}}\frac{Mm}{(M+m)^{\frac{1}{2}}}c^{\frac{1}{2}}$$

By the theory of the tides of a viscous spheroid (Phil. Trans., Part I., 1879, p. 13)

$$\tan 2\mathfrak{f}=\frac{2(n-\Omega)}{2\mathfrak{p}},\text{ where } 2\mathfrak{p}=\frac{2gaw}{19v}$$

Hence

$$\sin 4\mathfrak{f}=\frac{2(n-\Omega)/\mathfrak{p}}{1+(n-\Omega)^2/\mathfrak{p}^2},\text{ also } \tau^2=\left(\tfrac{3}{2}\right)^2\frac{\mu^2m^2}{c^6}$$

Hence (1) becomes

$$\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}\frac{dc^{\frac{1}{2}}}{dt}=\left(\tfrac{3}{2}\right)^2\frac{C}{\mathfrak{g}}\frac{(\mu m)^2}{c^6}\frac{(n-\Omega)/\mathfrak{p}}{1+(n-\Omega)^2/\mathfrak{p}^2}\,. \qquad (2)$$

Now let Ch be the angular momentum of the whole system, namely that due to the

* Phil. Trans., Part II., 1879, p. 459.

planet's rotation and to the orbital motion of all the satellites. And let CE be the whole energy, both kinetic and potential, of the system. Then h is the angular velocity with which the planet would have to rotate in order that the rotational momentum might be equal to that of the whole system; and E is twice the square of the angular velocity with which the planet would have to rotate in order that the kinetic energy of planetary rotation might be equal to the whole energy of the system. By the principle of conservation of momentum h is constant, and since the system is non-conservative of energy E is variable, and must diminish with the time.

The kinetic energy of the orbital motion of the satellite m is $\frac{1}{2}\mu Mm/c$, and the potential energy of position of the planet and satellite is $-\mu Mm/c$; the kinetic energy of the planet's rotation is $\frac{1}{2}Cn^2$. Thus we have,

$$Ch = Cn + \Sigma\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}c^{\frac{1}{2}} \quad \ldots \ldots \ldots (3)$$

$$2CE = Cn^2 - \Sigma\frac{\mu Mm}{c} \quad \ldots \ldots \ldots (4)$$

In the equations (3) and (4) we may regard C as a constant, provided we neglect the change of ellipticity of the planet's figure as its rotation slackens.

Let the symbol δ indicate partial differentiation; then from (3) and (4)

$$\frac{\delta n}{\delta(c^{\frac{1}{2}})} = -\frac{1}{C}\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}$$

$$-\frac{\delta E}{\delta(c^{\frac{1}{2}})} = \frac{1}{C}\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}n - \frac{1}{C}\frac{\mu Mm}{c^{\frac{3}{2}}}$$

But

$$\frac{1}{C}\frac{\mu Mm}{c^{\frac{3}{2}}} = \frac{1}{C}\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}\Omega$$

and therefore

$$-\frac{C(M+m)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}Mm}\frac{\delta E}{\delta(c^{\frac{1}{2}})} = n - \Omega \quad \ldots \ldots \ldots (5)$$

From equations (2) and (5) we may express the rate of increase of the square root of any satellite's distance in terms of the energy of the whole system, in the general case where the planet has any degree of viscosity. A good many transformations, analogous to those below, may be made in this general case, but as I shall only examine in detail the special case in which the viscosity is small, it will be convenient to make the transition thereto at once.

When the viscosity is small, $\mathfrak{p}$, which varies inversely as the viscosity, is large. Then, unless $n-\Omega$ be very large, $(n-\Omega)/\mathfrak{p}$ is small compared with unity. Thus in (2) we may neglect $(n-\Omega)^2/\mathfrak{p}^2$ in the denominator compared with unity.

Substituting from (5) in (2), and making this approximation, we have

$$\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}\frac{dc^{\frac{1}{2}}}{dt} = -\left(\tfrac{3}{2}\right)^2\frac{C}{\mathfrak{g}}\frac{(m\mu)^2}{c^6}\frac{C(M+m)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}Mm\mathfrak{p}}\frac{\delta E}{\delta(c^{\frac{1}{2}})} \quad \ldots \ldots (6)$$

Now let

$$\xi=\left(\frac{M}{M+m}\right)^{\frac{1}{2}}\left(\frac{c}{\alpha}\right)^{\frac{7}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where α is any constant length, which it may be convenient to take either as equal to the mean radius of the planet, or as the distance of some one of the satellites at some fixed epoch. ξ is different for each satellite and is subject to the suffixes 1, 2, 3, &c.

The equation (6) may be written

$$\left(\frac{M}{M+m}\right)^{\frac{1}{2}}7c^3\frac{dc^{\frac{1}{2}}}{dt}=-\left(\tfrac{3}{2}\right)^2\times 49\times\frac{\mu C^2}{M\mathfrak{g}\mathfrak{p}}\times\left(\frac{M+m}{M}\right)^{\frac{1}{2}}\frac{1}{7c^3}\,\frac{\delta E}{\delta(c^{\frac{1}{2}})}$$

Now let A

$$A=\left(\tfrac{3}{2}\right)^2\times 49\times\frac{\mu C^2}{M\alpha^7\mathfrak{g}\mathfrak{p}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

And we have

$$\frac{d\xi}{dt}=-A\frac{\delta E}{\delta\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

[In order to calculate A it may be convenient to develop its expression further.

$$M\mathfrak{g}\alpha^7=\tfrac{2}{5}\frac{\mu M}{a^3}\cdot M\alpha^7=\tfrac{5}{2}\mu C^2\left(\frac{\alpha}{a}\right)^7, \text{ so that } \frac{\mu C^2}{M\mathfrak{g}\alpha^7}=\tfrac{2}{5}\left(\frac{a}{\alpha}\right)^7$$

and

$$A=\left(\tfrac{3}{2}\right)^2\left(\tfrac{2}{5}\right)49\frac{(a/\alpha)^7}{\mathfrak{p}}, \text{ where } \mathfrak{p}=\frac{gaw}{19v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Since $\mathfrak{p}$ is an angular velocity A is a period of time, and A is the same for all the satellites.]

In (9) ξ is the variable, but it will be convenient to introduce an auxiliary variable x, such that

$$\left.\begin{aligned} x^7&=\xi \\ \text{or}\qquad x&=\left(\frac{M}{M+m}\right)^{\frac{1}{14}}\left(\frac{c}{\alpha}\right)^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Then

$$\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}\,c^{\frac{1}{2}}=\frac{\mu^{\frac{1}{2}}M^{\frac{13}{14}}m\alpha^{\frac{1}{2}}}{(M+m)^{\frac{3}{7}}}\,x$$

Let

$$\kappa=\frac{\mu^{\frac{1}{2}}M^{\frac{13}{14}}m\alpha^{\frac{1}{2}}}{C(M+m)^{\frac{3}{7}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

κ is different for each satellite and is subject to suffixes 1, 2, 3, &c.

Thus (3) may be written

$$h=n+\Sigma\kappa x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Again

$$\frac{\mu Mm}{c}=\frac{\mu M^{\frac{8}{7}}m}{(M+m)^{\frac{1}{7}}\alpha}\,\frac{1}{x^2}$$

Let $$\lambda=\frac{\mu M^{\frac{8}{7}} m}{C(M+m)^{\frac{1}{7}}\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

λ is different for each satellite and is subject to suffixes 1, 2, 3, &c.

On comparing (12) and (14) we see that

$$\frac{\lambda}{\kappa}=\Omega x^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This is of course merely a form of writing the equation

$$\mu(M+m)=\Omega^2 c^3$$

Then (4) may be written

$$2E=n^2-\Sigma\frac{\lambda}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

[In order to compute κ and λ we may pursue two different methods.

First, suppose $\alpha=a$, the planet's mean radius.

Then $$\frac{m\alpha^{\frac{1}{2}}}{C}=\frac{5}{2\nu a^{\frac{3}{2}}};\; M^{\frac{7}{14}}\mu^{\frac{1}{2}}=(ga^2)^{\frac{1}{2}};\; \frac{M^{\frac{6}{14}}}{(M+m)^{\frac{3}{7}}}=\left(\frac{\nu}{1+\nu}\right)^{\frac{3}{7}}$$

$\kappa=\frac{5}{2}[\nu^4(1+\nu)^3]^{-\frac{1}{7}}\left(\frac{g}{a}\right)^{\frac{1}{2}}$, of same dimensions as an angular velocity.

$\lambda=\frac{5}{2}[\nu^6(1+\nu)]^{-\frac{1}{7}}\left(\frac{g}{a}\right)$, of same dimensions as the square of an angular velocity.

If ν be large compared with unity, as is generally the case, the expressions become

$$\kappa=\frac{5m}{2M}\sqrt{\frac{g}{a}}, \qquad \lambda=\frac{5m}{2M}\left(\frac{g}{a}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Secondly, suppose M large compared with all the m's, and suppose for example that the solar system as a whole is the subject of investigation. Then take α as the earth's present radius vector, and ω as its present mean motion, and

$$\kappa=\frac{m}{C}\sqrt{\mu M\alpha}, \quad \text{and } \lambda=\frac{m}{C}\frac{\mu Mm}{\alpha}$$

$$\text{or} \quad \kappa=m\left(\frac{\omega\alpha^2}{C}\right), \qquad \lambda=m\left(\frac{\omega^2\alpha^2}{C}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

C is here the sun's moment of inertia.]

Then collecting results from (9), (13), (16), the equations which determine the changes in the system are

$$\left.\begin{array}{l} \dfrac{d\xi}{dt}=-A\dfrac{\delta E}{\delta\xi} \\ \text{and a similar equation for each satellite} \\ n=h-\Sigma\kappa x \\ 2E=n^2-\Sigma\dfrac{\lambda}{x^2} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where $x^7=\xi$; A is a certain time to be computed as above shown in (10); κ an angular velocity to be computed as above shown in (17) and (18); and λ the square of an angular velocity to be computed as above in (17) and (18).

Also
$$\xi=\left(\frac{M}{M+m}\right)^{\frac{1}{2}}\left(\frac{c}{a}\right)^{\frac{7}{2}}=\left(\frac{\nu}{1+\nu}\right)^{\frac{1}{2}}\left(\frac{c}{a}\right)^{\frac{7}{2}}$$

If ν be large compared with unity, ξ is very approximately proportional to the seventh power of the square root of the satellite's distance.

The solution of this system of simultaneous differential equations would give each of the ξ's in terms of the time; afterwards we might obtain n and E in terms of the time from the last two of (19).

These differential equations possess a remarkable analogy with those which represent HAMILTON's principle of varying action (THOMSON and TAIT's 'Nat. Phil.,' 1879, § 330 (14)).

The rate of loss of energy of the system may be put into a very simple form. This function has been called by Lord RAYLEIGH ('Theory of Sound,' vol. i., § 81) the Dissipation Function,* and the name is useful, because this function plays an important part in non-conservative systems.

In the present problem the Dissipation Function or Dissipativity is $-C\frac{dE}{dt}$.

Now
$$\frac{dE}{dt}=\Sigma\frac{\delta E}{\delta\xi}\frac{d\xi}{dt}$$

From (19) the dissipativity is therefore either

$$CA\Sigma\left(\frac{\delta E}{\delta\xi}\right)^2 \text{ or } \frac{C}{A}\Sigma\left(\frac{d\xi}{dt}\right)^2$$

This quantity is of course essentially positive.

It is easy to show that $\frac{\delta E}{\delta\xi}=-\frac{\kappa}{7x^6}(n-\Omega)$

Then on substituting for the various symbols in the expression for the dissipativity their values in terms of the original notation, we have

$$-\frac{dE}{dt}=\Sigma\frac{\tau^2}{\mathfrak{gp}}(n-\Omega)^2$$

Or if N be the tidal frictional couple corresponding to the satellite m,

$$-C\frac{dE}{dt}=\Sigma N(n-\Omega)$$

This last result would be equally true whatever were the viscosity of the planetary spheroid.

* Sir W. THOMSON prefers to modify the name by calling it Dissipativity.

The dissipativity, converted into heat by JOULE's equivalent, expresses the amount of heat generated per unit time within the planetary spheroid. This result has been already obtained in a different manner for the case of a single satellite in a previous paper ("Problems, &c.," Phil. Trans., Part II., 1879, p. 557).

§ 3. *Sketch of method for solution of the equations by series.*

It does not seem easy to obtain a rigorous analytical solution of the system (19) of differential equations. I have however solved the equations by series, so as to obtain analytical expressions for the ξ's, as far as the fourth power of the time. This solution is not well adapted for the purposes of the present paper, because the series are not rapidly convergent, and therefore cannot express those large changes in the configuration of the system which it is the object of the present paper to trace.

As no subsequent use is made of this solution, and as the analysis is rather long, I will only sketch the method pursued.

If $\frac{1}{49}A$ be taken as the unit of time $\frac{dE}{dt}=-\frac{1}{49}\Sigma\left(\frac{\delta E}{\delta\xi}\right)^2$

Differentiating again and again with regard to the time, and making continued use of this equation, we find d^2E/dt^2, d^3E/dt^3, &c., in terms of $\delta E/\delta\xi$.

It is then necessary to develop these expressions by performing the differentiations with regard to ξ.

An abridged notation was used in which $\left[\frac{\mathrm{a},\ \mathrm{b}}{\mathrm{p}}\right]^{\mathrm{k}}$ represented $\left(\frac{\mathrm{a}\lambda x^{-3}-\mathrm{b}n\kappa}{x^{\mathrm{p}}}\right)^{\mathrm{k}}$ or $\left[\frac{\kappa(\mathrm{a}\Omega-\mathrm{b}n)}{x^{\mathrm{p}}}\right]^{\mathrm{k}}$. With this notation the whole operation may be shown to depend on the performance of $\delta/\delta\xi$ on expressions of the form

$$\Sigma\gamma\left[\frac{\mathrm{a}_1,\ \mathrm{b}_1}{\mathrm{p}_1}\right]^{\mathrm{k}_1}\left[\frac{\mathrm{a}_2,\ \mathrm{b}_2}{\mathrm{p}_2}\right]^{\mathrm{k}_2}\cdots\left[\frac{\mathrm{a}_r,\ \mathrm{b}_r}{\mathrm{p}_r}\right]^{\mathrm{k}_r}$$

where γ is independent of ξ, but may be a function of the mass of each satellite.

Having evaluated the successive differentials of E we have

$$E=E_0+t\left(\frac{dE}{dt}\right)_0+\frac{t^2}{1.2}\left(\frac{d^2E}{dt^2}\right)_0+\frac{t^3}{1.2.3}\left(\frac{d^3E}{dt^3}\right)_0+\text{ \&c.}$$

Where the suffix 0 indicates that the value, corresponding to $t=0$, is to be taken.

It is also necessary to evaluate the successive differentials of $\delta E/\delta\xi$ with regard to the time, and then we have

$$\xi=\xi_0-t\left(\frac{\delta E}{\delta\xi}\right)_0-\frac{t^2}{1.2}\left(\frac{\delta}{\delta\xi}\frac{dE}{dt}\right)_0-\frac{t^3}{1.2.3}\left(\frac{\delta}{\delta\xi}\frac{d^2E}{dt^2}\right)_0-\text{ \&c.}$$

The coefficient of t^4 was found to be very long even with the abridged notation, and involved squares and products of Σ's.

§ 4. *Graphical solution in the case when there are not more than three satellites.*

Although a general analytical solution does not seem attainable, yet the equations have a geometrical or quasi-geometrical meaning, which makes a complete graphical solution possible, at least in the case where there are not more than three satellites.

To explain this I take the case of two satellites only, and to keep the geometrical method in view I change the notation, and write z for E, x for ξ_1, and y for ξ_2, also I write Ω_x for Ω_1, and Ω_y for Ω_2. The unit of time is chosen so that $A=1$.

Then the equations (19) become

$$\frac{dx}{dt}=-\frac{\partial z}{\partial x}, \quad \frac{dy}{dt}=-\frac{\partial z}{\partial y} \qquad (20)$$

and

$$2z=(h-\kappa_1 x^{\frac{1}{7}}-\kappa_2 y^{\frac{1}{7}})^2-\frac{\lambda_1}{x^{\frac{2}{7}}}-\frac{\lambda_2}{y^{\frac{2}{7}}} \qquad (21)$$

Now suppose a surface constructed to illustrate (21), x, y, z being the coordinates of any point on it. Let the axes of x and y be drawn horizontally, and that of z vertically upwards. The z ordinate of course gives the energy of the system corresponding to any values of x and y which are consistent with the given angular momentum h.

We have for the dissipativity of the system

$$-\frac{dz}{dt}=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Whence

$$\frac{dx}{dz}=\frac{\frac{\partial z}{\partial x}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}, \quad \frac{dy}{dz}=\frac{\frac{\partial z}{\partial y}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} \qquad (22)$$

Let $(X-x)/\lambda=(Y-y)/\mu=Z-z$ be the equations to a straight line through a point x, y, z on the surface. Then if this line lies in the tangent plane at that point

$$\lambda\frac{\partial z}{\partial x}+\mu\frac{\partial z}{\partial y}-1=0.$$

The inclination of this line to the axis of z will be a maximum or minimum when $\lambda^2+\mu^2$ is a maximum or minimum. In other words if this straight line is a tangent line to the steepest path through x, y, z on the surface, $\lambda^2+\mu^2$ must be a maximum or minimum.

Hence for this condition to be fulfilled we must have

$$\lambda\delta\lambda+\mu\delta\mu=0$$

$$\frac{\partial z}{\partial x}\delta\lambda+\frac{\partial z}{\partial y}\delta\mu=0$$

And therefore $\lambda\Big/\frac{\partial z}{\partial x}=\mu\Big/\frac{\partial z}{\partial y}$, and these are both equal to $1\Big/\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}$

Therefore the equation to the tangent to the steepest path is

$$\frac{X-x}{\delta z/\delta x}=\frac{Y-y}{\delta z/\delta y}=\frac{Z-z}{(\delta z/\delta x)^2+(\delta z/\delta y)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If this steepest path on the energy surface is the path actually pursued by the point which represents the configuration of the system, equation (23) must be satisfied by

$$X=x+\frac{dx}{dz}\delta z, \quad Y=y+\frac{dy}{dz}\delta z, \quad Z=z+\delta z$$

And therefore we must have

$$\frac{dx}{dz}=\frac{\dfrac{\delta z}{\delta x}}{\left(\dfrac{\delta z}{\delta x}\right)^2+\left(\dfrac{\delta z}{\delta y}\right)^2}, \quad \frac{dy}{dz}=\frac{\dfrac{\delta z}{\delta y}}{\left(\dfrac{\delta z}{\delta x}\right)^2+\left(\dfrac{\delta z}{\delta y}\right)^2}$$

But these are the values already found in (22) for dx/dz and dy/dz.

Therefore we conclude that the representative point always slides down a steepest path on the energy surface. Hence it only remains to draw the surface, and to mark out the lines of steepest slope in order to obtain a complete graphical solution of the problem. Since the lines of greatest slope cut the contours at right angles, if we project the contours orthogonally on to the plane of $x\,y$, and draw the system of orthogonal trajectories of the contours, we obtain a solution in two dimensions. This solution will be exhibited below, but for the present I pass on to more general considerations.

A precisely similar argument might be applied to the case where there are any number of satellites, only as space has but three dimensions, a geometrical solution is not possible. If there be r satellites, then the problem to be solved may be stated in geometrical language thus :—

It is required to find the path which is inclined at the least angle to the axis of E on the locus

$$2E=(h-\Sigma\kappa\xi^{\frac{1}{7}})^2-\Sigma\frac{\lambda}{\xi^{\frac{2}{7}}}$$

This locus is described in space of $r+1$ dimensions. One axis is that of E, and the remaining r axes are the axes of the r different ξ's. The solution may be depressed so as to merely require space of r dimensions, for we may, in space of r dimensions, construct the orthogonal trajectories of the contour loci found by attributing various values to E.

Thus we might actually solve geometrically the case of three satellites. The energy locus here involves space of four dimensions, but the contour loci are a family of surfaces in three dimensions. If such a system of surfaces were actually constructed, it would be possible to pass through them a number of wires or threads which should be a good approximation to the orthogonal trajectories. The trouble of execution would however be hardly repaid by the results, because most of the interesting general conclusions may be drawn from the case of two satellites, where we have only to deal with curves.

If the case of a single satellite be considered, we see that the energy locus is a curve, and the transit along the steepest path degenerates merely into travelling down hill. Now as the slopes of the energy curve are not altered in direction, but merely in steepness, by taking the abscissas of points on the curve as any power of ξ, the solution may still be obtained if we take x (or $\xi^{\frac{1}{7}}$) as the abscissa instead of ξ. This reduces the solution to exactly that which was given in a previous paper, where the graphical method was applied to the case of a single satellite.*

§ 5. *The graphical method in the case of two satellites.*

I now return to the special case in which there are only two satellites. The equation to the surface of energy is given in (21). The maxima and minima values of z (if any) are given by equating $\partial z/\partial x$ and $\partial z/\partial y$ to zero. This gives

$$\left.\begin{aligned} h-\kappa_1 x^{\frac{1}{7}}-\kappa_2 y^{\frac{1}{7}} &= \frac{\lambda_1}{\kappa_1}\frac{1}{x^{\frac{3}{7}}} \\ h-\kappa_1 x^{\frac{1}{7}}-\kappa_2 y^{\frac{1}{7}} &= \frac{\lambda_2}{\kappa_2}\frac{1}{y^{\frac{3}{7}}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

By (15) and (19) we see that these equations may be written

$$\left.\begin{aligned} n &= \Omega_x \\ n &= \Omega_y \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

They also lead to the equations

$$\left.\begin{aligned} (x^{\frac{1}{7}})^4-\frac{h-\kappa_2 y^{\frac{1}{7}}}{\kappa_1}(x^{\frac{1}{7}})^3+\frac{\lambda_1}{\kappa_1^{\,2}} &= 0 \\ (y^{\frac{1}{7}})^4-\frac{h-\kappa_1 x^{\frac{1}{7}}}{\kappa_2}(y^{\frac{1}{7}})^3+\frac{\lambda_2}{\kappa_2^{\,2}} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Now an equation of the form $Y^4-\alpha Y^3+\beta=0$ may be written $(Y\beta^{-\frac{1}{4}})^4-\alpha\beta^{-\frac{3}{4}}(Y\beta^{-\frac{1}{4}})^3+1=0$. And I have proved in a previous paper † that an equation $x^4-hx^3+1=0$ has two real roots, if h be greater than $4/3^{\frac{3}{4}}$, but has no real roots if h be less than $4/3^{\frac{3}{4}}$. Hence it follows that this equation in Y has two real roots, if α be greater than $4\beta^{\frac{1}{4}}/3^{\frac{3}{4}}$, but no real roots if it be less.

Then if we consider the two equations (26) as biquadratics for $x^{\frac{1}{7}}$ and $y^{\frac{1}{7}}$ respectively, we see that the first has, or has not, a pair of real roots, according as

* Proc. Roy. Soc., No. 197, 1879.

† Ibid., No. 202, 1880, p. 260–263.

$$h-\kappa_2 y^{\frac{1}{7}} \text{ is greater or less than } \left(\frac{4}{3^{\frac{3}{4}}}\right) \lambda_1^{\frac{1}{4}}\kappa_1^{\frac{1}{2}},$$

and the second has, or has not, a pair of real roots, according as

$$h-\kappa_1 x^{\frac{1}{7}} \text{ is greater or less than } \left(\frac{4}{3^{\frac{3}{4}}}\right) \lambda_2^{\frac{1}{4}}\kappa_2^{\frac{1}{2}}.$$

Now if we substitute for the λ's and κ's their values, we find that

$$\lambda_1^{\frac{1}{4}}\kappa_1^{\frac{1}{2}}=\frac{\mu^{\frac{1}{2}}(Mm_1)^{\frac{3}{4}}}{C^{\frac{3}{4}}(M+m_1)^{\frac{1}{4}}}$$

$$\lambda_2^{\frac{1}{4}}\kappa_2^{\frac{1}{2}}=\text{the same with } m_2 \text{ in place of } m_1.$$

Now let γ_1 and γ_2 be two lengths determined by the equations

$$\frac{Mm_1}{M+m_1}\gamma_1^2=\frac{Mm_2}{M+m_2}\gamma_2^2=C.$$

Or in words—let γ_1 be such a distance that the moment of inertia of the planet (concentrated at its centre) and the first satellite about their common centre of inertia may be equal to the planet's moment of inertia about its axis of rotation; and let γ_2 be a similar distance involving the second satellite instead of the first.

And let ω_1, ω_2 be two angular velocities determined by the equations

$$\omega_1^2\gamma_1^3=\mu(M+m_1), \qquad \omega_2^2\gamma_2^3=\mu(M+m_2).$$

Or in words—let ω_1 be the angular velocity of the first satellite when revolving in a circular orbit at distance γ_1, and ω_2 a similar angular velocity for the second satellite when revolving at distance γ_2.

Now
$$\left[\frac{\mu Mm_1}{C}\right]^{\frac{1}{2}}=\omega_1\gamma_1^{\frac{1}{2}} \text{ and } \left[\frac{Mm_1}{C(M+m_1)}\right]^{\frac{1}{4}}=\frac{1}{C}\frac{Mm_1}{M+m_1}\gamma_1^{\frac{3}{2}}.$$

So that
$$\lambda_1^{\frac{1}{4}}\kappa_1^{\frac{1}{2}}=\frac{1}{C}\frac{Mm_1}{M+m_1}\omega_1\gamma_1^2,$$

and similarly $\lambda_2^{\frac{1}{4}}\kappa_2^{\frac{1}{2}}$=the same with the suffix 2 in place of 1. Hence the first of the two equations (26) has, or has not, a pair of real roots, according as

$$C(h-\kappa_2 y^{\frac{1}{7}}) \text{ is greater or less than } \frac{4}{3^{\frac{3}{4}}}\frac{Mm_1}{M+m_1}\omega_1\gamma_1^2,$$

and the second has, or has not, a pair of real roots, according as

$$C(h-\kappa_1 x^{\frac{1}{7}}) \text{ is greater or less than } \frac{4}{3^{\frac{3}{4}}}\frac{Mm_2}{M+m_2}\omega_2\gamma_2^2.$$

It is obvious that $Mm_1\omega_1\gamma_1^2/(M+m_1)$ is the orbital momentum of the first satellite when revolving at distance γ_1, and similarly $Mm_2\omega_2\gamma_2^2/(M+m_2)$ is the orbital momentum of the second satellite when revolving at distance γ_2.

If the second or y-satellite be larger than the first or x-satellite the latter of these momenta is larger than the first.

Now Ch is the whole angular momentum of the system, and in order that there may be maxima and minima determined by the equations $\partial z/\partial x=0$, $\partial z/\partial y=0$, the equations (26) must have real roots. Then on putting y equal to zero in the first of the above conditions, and x equal to zero in the second we get the following results:—

First, there are no maxima and minima points for sections of the energy surface either parallel to x or y, if the whole momentum of the system be less than $4/3^{\frac{3}{4}}$ times the orbital momentum of the smaller or x-satellite when moving at distance γ_1.

Second, there are maxima and minima points for sections parallel to x, but not for sections parallel to y, if the whole momentum be greater than $4/3^{\frac{3}{4}}$ times the orbital momentum of the smaller or x-satellite when moving at distance γ_1, but less than $4/3^{\frac{3}{4}}$ times the orbital momentum of the larger or y-satellite when moving at distance γ_2.

Third, there are maxima and minima for both sections, if the whole momentum be greater than $4/3^{\frac{3}{4}}$ times the orbital momentum of the larger or y-satellite when moving at distance γ_2.

This third case now requires further subdivision, according as whether there are not or are absolute maximum or minimum points on the surface.

If there are such points the two equations (24) or (25) must be simultaneously satisfied.

Hence we must have $n=\Omega_x=\Omega_y$, in order that there may be a maximum or minimum point on the surface.

But in this case the two satellites revolve in the same periodic time, and may be deemed to be rigidly connected together, and also rigidly connected with the planet. Hence the configurations of maximum or minimum energy are such that all three bodies move as though rigidly connected together.

The simultaneous satisfaction of (24) necessitates that

$$x^{\frac{3}{7}}=\frac{\lambda_1\kappa_2}{\kappa_1\lambda_2}y^{\frac{3}{7}} \text{ or } y^{\frac{3}{7}}=\frac{\lambda_2\kappa_1}{\kappa_2\lambda_1}x^{\frac{3}{7}}$$

Hence the equations (24) become

$$h-\left[\kappa_1+\kappa_2\left(\frac{\lambda_2\kappa_1}{\lambda_1\kappa_2}\right)^{\frac{1}{3}}\right]x^{\frac{1}{7}}=\frac{\lambda_1}{\kappa_1}\,\frac{1}{x^{\frac{3}{7}}}$$

$$h-\left[\kappa_1\left(\frac{\lambda_1\kappa_2}{\lambda_2\kappa_1}\right)^{\frac{1}{3}}+\kappa_2\right]y^{\frac{1}{7}}=\frac{\lambda_2}{\kappa_2}\,\frac{1}{y^{\frac{3}{7}}}$$

These equations may be written

$$\left.\begin{aligned}(x^{\frac{1}{7}})^4-\frac{h}{\kappa_1+\kappa_2(\lambda_2\kappa_1/\lambda_1\kappa_2)^{\frac{1}{3}}}(x^{\frac{1}{7}})^3+\frac{\lambda_1/\kappa_1}{\kappa_1+\kappa_2(\lambda_2\kappa_1/\lambda_1\kappa_2)^{\frac{1}{3}}}=0\\(y^{\frac{1}{7}})^4-\frac{h}{\kappa_1(\lambda_1\kappa_2/\lambda_2\kappa_1)^{\frac{1}{3}}+\kappa_2}(y^{\frac{1}{7}})^3+\frac{\lambda_2/\kappa_2}{\kappa_1(\lambda_1\kappa_2/\lambda_2\kappa_1)^{\frac{1}{3}}+\kappa_2}=0\end{aligned}\right\}\quad\ldots\quad(27)$$

Then treating these biquadratics in the same way as before, we find that they have, or have not, two real roots, according as h is greater or less than

$$\frac{4}{3^{\frac{3}{4}}}\left[(\lambda_1\kappa_1{}^2)^{\frac{1}{3}}+(\lambda_2\kappa_2{}^2)^{\frac{1}{3}}\right]^{\frac{3}{4}}$$

Now
$$(\lambda_1\kappa_1{}^2)^{\frac{1}{3}}+(\lambda_2\kappa_2{}^2)^{\frac{1}{3}}=\frac{\mu^{\frac{2}{3}}M}{C}\left[\frac{m_1}{(M+m_1)^{\frac{1}{3}}}+\frac{m_2}{(M+m_2)^{\frac{1}{3}}}\right]$$

Therefore there is, or there is not, a pair of real solutions of the equations $n=\Omega_x=\Omega_y$, according as the total momentum of the system is, or is not, greater than

$$\frac{4}{3^{\frac{3}{4}}}\,\mu^{\frac{1}{2}}C^{\frac{1}{4}}M^{\frac{3}{4}}\left[\frac{m_1}{(M+m_1)^{\frac{1}{3}}}+\frac{m_2}{(M+m_2)^{\frac{1}{3}}}\right]^{\frac{3}{4}}$$

And this is also the criterion whether or not there is a maximum, or minimum, or maximum-minimum point on the energy surface.

In the case where the masses of the satellites are small compared with the mass of the planet, we may express the critical value of the momentum of the system in the form

$$\frac{4}{3^{\frac{3}{4}}}\,\mu^{\frac{1}{2}}C^{\frac{1}{4}}\frac{[M(m_1+m_2)]^{\frac{3}{4}}}{(M+m_1+m_2)^{\frac{1}{4}}}$$

A comparison of this critical value with the two previous ones shows that if the two satellites be fused together, and if γ be such that

$$\frac{M(m_1+m_2)}{M+m_1+m_2}\gamma^2=C,$$

and if ω be the orbital angular velocity of the compound satellite when moving at distance γ, then the above critical value of the momentum of the whole system is

$$\frac{4}{3^{\frac{3}{4}}}\frac{M(m_1+m_2)}{M+m_1+m_2}\omega\gamma^2$$

and this is $4/3^{\frac{3}{4}}$ times the orbital momentum of the compound satellite when revolving at distance γ.

Hence if the masses of the satellites are small compared with that of the planet, there are, or are not, maximum or minimum or maximum-minimum points on the surface of energy, according as the total momentum of the system is greater or less than $4/3^{\frac{3}{4}}$ times the orbital momentum of the compound satellite when moving at distance γ.

In the case where the masses of the satellites are not small compared with that of the planet, I leave the criterion in its analytical form.

There are thus three critical values of the momentum of the whole system, and the actual value of the momentum determines the character of the surface of energy according to its position with reference to these critical values.

In proceeding to consider the graphical method of solution by means of the contour lines of the energy surface, I shall choose the total momentum of the system to be greater than this third critical value, and the surface will have a maximum point. From the nature of the surface in this case we shall be able to see how it would differ if the total momentum bore any other position with reference to the three critical values. It will be sufficient if we only consider the case where the masses of the two satellites are small compared with that of the planet.

By (17) we have, with an easily intelligible alternative notation,

$$\left.\begin{matrix}\kappa_1\\ \kappa_2\end{matrix}\right\}=\frac{5}{2}\frac{\left.\begin{matrix}m_1\\ m_2\end{matrix}\right\}}{M}\sqrt{\frac{g}{a}},\quad \left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\}=\frac{5}{2}\frac{\left.\begin{matrix}m_1\\ m_2\end{matrix}\right\}}{M}\frac{g}{a}$$

Now κ_1 is an angular velocity, and if we choose $1/\kappa_1$ as the unit of time, we have

$$\kappa_1=1,\ \kappa_2=\frac{m_2}{m_1}$$

also

$$\lambda_1=\tfrac{2}{5}\frac{M}{m_1}\kappa_1^2\quad \lambda_2=\tfrac{2}{5}\frac{M}{m_2}\kappa_2^2$$

Now if we choose the mass of the first satellite as unit of mass, then $m_1=1$, and we have

$$\kappa_1=1,\ \kappa_2=m_2,\ \lambda_1=\tfrac{2}{5}M,\ \lambda_2=\tfrac{2}{5}Mm_2$$

The unit of length has been already chosen as equal to the mean radius of the planet.

Then substituting in (21) we have as the equation to the energy surface

$$2z=(h-x^{\frac{1}{7}}-m_2y^{\frac{1}{7}})^2-\tfrac{2}{5}M\left(\frac{1}{x^{\frac{2}{7}}}+\frac{m_2}{y^{\frac{2}{7}}}\right)$$

And since we suppose m_1 and m_2 to be small compared with M, we have

$$x=\left(\frac{c_1}{a}\right)^{\frac{7}{2}},\ y=\left(\frac{c_2}{a}\right)^{\frac{7}{2}}$$

On account of the abruptness of the curvatures, this surface is extremely difficult to illustrate unless the figure be of very large size, and it is therefore difficult to choose appropriate values of h, M, m_2, so as to bring the figure within a moderate compass.

In order to exhibit the influence of unequal masses in the satellites, I choose $m_2=2$, the mass of the first satellite being unity. I take $M=50$, so that $\frac{2}{5}M=20$.

Then with these values for M and m, the first critical value for h is 3·711, the second is 6·241, and the third is 8·459.

I accordingly take $h=9$, which is greater than the third critical value. The surface to be illustrated then has the equation

$$2z=(9-x^{\frac{1}{7}}-2y^{\frac{1}{7}})^2-20\left(\frac{1}{x^{\frac{2}{7}}}+\frac{2}{y^{\frac{2}{7}}}\right)$$

There is also another surface to be considered, namely

$$n = 9 - x^{\frac{1}{7}} - 2y^{\frac{1}{7}}$$

which gives the rotation of the planet corresponding to any values of x and y.

The equations

$$n = \Omega_x, \quad n = \Omega_y$$

have also to be exhibited.

The computations requisite for the illustration were laborious, as I had to calculate values of z and n corresponding to a large number of values of x and y, and then by graphical interpolation to find the values of x and y, corresponding to exact values of z and n.

The surface of energy will be considered first.

Plate 61 shows the contour-lines (that is to say, lines of equal energy) in the positive quadrant, z being either positive or negative.

I speak below as though the paper were held horizontally, and as though positive z were drawn vertically upwards.

The numbers written along the axes give the numerical values of x and y.

The numbers written along the curves are the corresponding values of $-2z$. Since the numbers happen to be all negative, smaller numbers indicate greater energy than larger ones; and, accordingly, in going down hill we pass from smaller to greater numbers.

The full-line contours are equidistant, and correspond to the values 9, $8\frac{1}{2}$, 8, $7\frac{1}{2}$, 7, $6\frac{1}{2}$, and 6 of $-2z$; but since the slopes of the surface are very gentle in the central part, dotted lines (. . . .) are drawn for the contours $7\frac{3}{4}$ and $7\frac{1}{4}$.

The points marked 5·529 and 7·442 are equidistant from x and y, and therefore correspond to the case when the two satellites have the same distance from the planet, or, which amounts to the same thing, are fused together. The former is a maximum point on the surface, the latter a maximum-minimum.

The dashed line (— — —) through 7·442 is the contour corresponding to that value of $-2z$.

The chain-dot lines (—·—·—) through the same point will be explained below.

An inspection of these contours shows that along the axes of x and y the surface has infinitely deep ravines; but the steepness of the cliffs diminishes as we recede from the origin.

The maximum point 5·529 is at the top of a hill bounded towards the ravines by very steep cliffs, but sloping more gradually in the other directions.

The maximum-minimum point 7·442 is on a saddle-shaped part of the surface, for we go up hill, whether proceeding towards O or away from O, and we go down hill in either direction perpendicular to the line towards O.

If the total angular momentum of the system had been less than the smallest critical value, the contour lines would all have been something like rectangular hyperbolas with the axes of x and y as asymptotes, like the outer curves marked 6, $6\frac{1}{2}$, 7 in Plate 61. In this case the whole surface would have sloped towards the axes.

If the momentum had been greater than the smallest, and less than the second critical value, the outer contours would have still been like rectangular hyperbolas, and the branches which run upwards, more or less parallel to y, would still have preserved that character nearer to the axes, whilst the branches more or less parallel to x would have had a curve of contrary reflexure, somewhat like that exhibited by the curve $7\frac{1}{2}$ in Plate 61, but less pronounced. In this case all the lines of steepest slope would approach the axis of x, but some of them in some part of their course would recede from the axis of y.

If the momentum had been greater than the second, but less than the third critical value, the contours would still all have been continuous curves, but for some of the inner ones there would have been contrary reflexure in both branches, somewhat like the curve marked $7\frac{1}{2}$ in Plate 61. There would still have been no closed curves amongst the contours. Here some of the lines of greatest slope would in part of their course have receded from the axis of x, and some from the axis of y, but the same line of greatest slope would never have receded from both axes.

Finally, if the momentum be greater than the third critical value, we have the case exhibited in Plate 61.

Plate 62, fig. 1, exhibits the lines of greatest slope on the surface. It was constructed by making a tracing of Plate 61, and then drawing by eye the orthogonal trajectories of the contours of equal energy. The dashed line (— — —) is the contour corresponding to the maximum-minimum point 7·442 of Plate 61. The chain-dot line (—·—·—) will be explained later.

One set of lines all radiate from the maximum point 5·529 of Plate 61. The arrows on the curves indicate the downward direction. It is easy to see how these lines would have differed, had the momentum of the system had various smaller values.

Plate 62, fig. 2, exhibits the contour lines of the surface

$$n=9-x^{\frac{1}{2}}-2y^{\frac{1}{2}}$$

It is drawn on the same scale as Plate 61 and Plate 62, fig. 1.

The computations for the energy surface, together with graphical interpolation, gave values of x and y corresponding to exact values of n.

The axis of n is perpendicular to the paper, and the numbers written on the curves indicate the various values of n.

These curves are *not* asymptotic to the axes, for they all cut both axes. The angles, however, at which they cut the axes are so acute that it is impossible to exhibit the intersections.

None of the curves meet the axis of x within the limits of the figure.

The curve $n=3$ meets the axis of y when $y=2150$, and that for $n=3\frac{1}{2}$ when $y=1200$, but for values of n smaller than 3 the intersections with the axis of y do not fall within the figure. The thickness, which it is necessary to give to the lines in drawing, obviously prevents the possibility of showing these facts, except in a figure of very large size.

On the side remote from the origin of the curve marked 0, n is negative, on the nearer side positive.

Since $M=50$,

$$\Omega_x=20/x^{\frac{3}{7}} \text{ and } \Omega_y=20/y^{\frac{3}{7}}$$

Hence the lines on the figure, for which Ω_x is constant, are parallel to the axis of y, and those for which Ω_y is constant are parallel to the axis of x.

The points are marked off along each axis for which Ω_x or Ω_y are equal to $3\frac{1}{2}$, 3, $2\frac{1}{2}$, 2, $1\frac{1}{2}$, 1. The points for which they are equal to $\frac{1}{2}$ fall outside the figure.

Now, if we draw parallels to y through these points on the axis of x, and parallels to x through the points on the axis of y, these parallels will intersect the n curves of the same magnitude in a series of points. For example, $\Omega_x=1\frac{1}{2}$, when x is about 420, and the parallel to y through this point intersects the curve $n=1\frac{1}{2}$, where y is about 740. Hence the first or x-satellite moves as a rigid body attached to the planet, when the first satellite has a distance $(420)^{\frac{2}{7}}$, and the second a distance $(740)^{\frac{2}{7}}$. In this manner we obtain a curve shown as chain-dot (—·—·—) and marked $\Omega_x=n$ for every point on which the first satellite moves as though rigidly connected with the planet; and similarly there is a second curve (—·—·—) marked $\Omega_y=n$ for every point on which the second satellite moves as though rigidly connected with the planet. This pair of curves divides space into four regions, which are marked out on the figure.

The space comprised between the two, for which Ω_x and Ω_y are both less than n, is the part which has most interest for actual planets and satellites, because the satellites of the solar system in general revolve slower than their planets rotate.

If the sun be left out of consideration, the Martian system is exemplified by the space $\Omega_x>n$, $\Omega_y<n$, because the smaller and inner satellite revolves quicker than the planet rotates, and the larger and outer one revolves slower.

The little quadrilateral space near O is of the same character as the external space $\Omega_x>n$, $\Omega_y>n$, but there is not room to write this on the figure.

These chain-dot curves are marked also on Plates 61 and 62, fig. 1. In Plate 61 the line $\Omega_x=n$ passes through all those points on the contours of energy whose tangents are parallel to x, and the line $\Omega_y=n$ passes through points whose tangents are parallel to y.

The tangents to the lines of greatest slope are perpendicular to the tangents to the contours of energy; hence in Plate 62, fig. 1, $\Omega_x=n$ passes through points whose tangents are parallel to y, and $\Omega_y=n$ through points whose tangents are parallel to x.

Within each of the four regions into which space is thus divided the lines of slope preserve the same character; so that if, for example, at any part of the region they are receding from x and y, they do so throughout.

This is correct, because dx/dt changes sign with $n-\Omega_x$ and dy/dt with $n-\Omega_y$; also either $n-\Omega_x$ or $n-\Omega_y$ changes sign in passing from one region to another. In these figures a line drawn at 45° to the axes through the origin divides the space into two parts; in the upper region y is greater than x, and in the lower x is greater than y. Hence configurations, for which the greater or y-satellite is exterior to the lesser or x-satellite, are represented by points in the upper space and those in which the lesser satellite is exterior by the lower space.

In the figures of which I have been speaking hitherto the abscissas and ordinates are the $\frac{7}{2}$ power of the distances of the two satellites; now this is an inconveniently high power, and it is not very easy to understand the physical meaning of the result. I have therefore prepared another figure in which the abscissas and ordinates are the actual distances. In Plate 63, fig. 3, the curves are no longer lines of steepest slope.

The reduction from Plate 62, fig. 1, to Plate 63, fig. 3, involved the raising of all the ordinates and abscissas of the former one to the $\frac{2}{7}$ power. This process was rather troublesome, and Plate 63, fig. 3, cannot claim to be drawn with rigorous accuracy; it is, however, sufficiently exact for the hypothetical case under consideration. If we had to treat any actual case, it would only be necessary to travel along a single line of change, and for that purpose special methods of approximation might be found for giving more accurate results.

In this figure the numbers written along the axes denote the distances of the satellites in mean radii of the planet—the radius of the planet having been chosen as the unit of length.

The chain-dot curves, as before, enclose the region for which the orbital angular velocities of the satellites are less than that of the planet's rotation. The line at 45° to the axes marks out the regions for which the larger satellite is exterior or interior to the smaller one.

Let us consider the closed space, within which Ω_x and Ω_y are less than n.

The corner of this space is the point of maximum energy, from which all the curves radiate.

Those curves which have tangents inclined at more than 45° to the axis of x denote that, during part of the changes, the larger satellite recedes more rapidly from the planet than the smaller one.

If the curve cuts the 45° line, it means that the larger satellite catches up the smaller one. Since these curves all pass from the lower to the upper part of the space, it follows that this will only take place when the larger satellite is initially interior. According to the figure, after catching up the smaller satellite, the larger satellite becomes exterior. In reality there would probably either be a collision or the pair of satellites would form a double system like the earth and moon. After this the smaller satellite becomes almost stationary, revolves for an instant as though rigidly connected with the planet, and then slower than the planet revolves (when the curve passes out of the closed space); the smaller satellite then falls into the planet, whilst the larger satellite maintains a sensibly constant distance from the planet.

If we take one of the other curves corresponding to the case of the larger satellite being interior, we see that the smaller satellite may at first recede more rapidly than the larger, and then the larger more rapidly than the smaller, but not so as to catch it up. The larger one then becomes nearly stationary, whilst the smaller one still recedes. The larger one then falls in, whilst the smaller one is nearly stationary.

If we now consider those curves which are from the beginning in the upper half of the closed space, we see that if the larger satellite is initially exterior, it recedes at first rapidly, whilst the smaller one recedes slowly. The smaller and inner satellite then comes to revolve as though rigidly connected with the planet, and afterwards falls into the planet, whilst the distance of the larger one remains nearly unaltered.

Either satellite comes into collision with the planet when its distance therefrom is unity. When this takes place the colliding satellite becomes fused with the planet, and the system becomes one where there is only a single satellite; this case might then be treated as in previous papers.

The divergences of the curves from the point of maximum energy shows that a very small difference of initial configuration in a pair of satellites may in time lead to very wide differences of configuration. Accordingly tidal friction alone will not tend to arrange satellites in any determinate order. It cannot, therefore, be definitely asserted that tidal friction has not operated to arrange satellites in any order which may be observed.

I have hitherto only considered the positive quadrant of the energy surface, in which both satellites revolve positively about the planet. There are, however, three other cases, viz.: where both revolve negatively (in which case the planet necessarily revolves positively, so as to make up the positive angular momentum), or where one revolves negatively and the other positively.

These cases will not be discussed at length, since they do not possess much interest.

Plate 63, fig. 4, exhibits the contours of energy for that quadrant in which the smaller or x-satellite revolves positively and the larger or y-satellite negatively. This figure may be conceived as joined on to Plate 61, so that the x-axes coincide. The numbers written on the contours are the values of $2z$; they are positive and pretty large. Whence it follows that these contours are enormously higher than those shown in Plate 61, where all the numbers on the contours were negative.

The contours explain the nature of the surface. It may, however, be well to remark that, although the contours appear to recede back from the x-axis for ever, this is not the case; for, after receding from the axis for a long way, they ultimately approach it again, and the axis is asymptotic to each of them. The point, at which the tangent to each contour is parallel to the axis of x, becomes more and more remote the higher the contour.

The lines of steepest slope on this surface give, as before, the solution of the problem.

If we hold this figure upside down, and read x for y and y for x, we get a figure which represents the general nature of the surface for the case where the x-satellite revolves negatively and the y-satellite positively. But of course the figure would not be drawn correctly to scale.

The contours for the remaining quadrant, in which both satellites revolve negatively would somewhat resemble a family of rectangular hyperbolas with the axes as asymptotes. I have not thought it worth while to construct them, but the physical interpretation is obviously that both satellites always must approach the planet.

II.

A DISCUSSION OF THE EFFECTS OF TIDAL FRICTION WITH REFERENCE TO THE EVOLUTION OF THE SOLAR SYSTEM.

§ 6. *General consideration of the problem presented by the solar system.*

In a series of previous papers I have traced out the changes in the manner of motion of the earth and moon which must have been caused by tidal friction. By adopting the hypothesis that tidal friction has been the most important element in the history of those bodies, we are led to coordinate together all the elements in their motions in a manner so remarkable, that the conclusion can hardly be avoided that the hypothesis contains a great amount of truth.

Under these circumstances it is natural to inquire whether the same agency may not have been equally important in the evolution of the other planetary sub-systems, and of the solar system as a whole.

This inquiry necessarily leads on to wide speculations, but I shall endeavour to derive as much guidance as possible from numerical data.

In the first part of the present paper the theory of the tidal friction of a planet, attended by several satellites, has been treated.

It would, at first sight, seem natural to replace this planet by the sun, and the satellites by the planets, and to obtain an approximate numerical solution. We might suppose that such a solution would afford indications as to whether tidal friction has or has not been a largely efficient cause in modifying the solar system.

The problem here suggested for solution differs, however, in certain points from that actually presented by the solar system, and it will now be shown that these differences are such as would render the solution of no avail.

The planets are not particles, as the suggested problem would suppose them to be, but they are rotating spheroids in which tides are being raised both by their own satellites and by the sun. They are, therefore, subject to a complicated tidal friction; the reaction of the tides raised by the satellites goes to expand the orbits of the satellites, but the reaction of the tide raised in the planet by the sun, and that raised in the sun by the planet both go towards expanding the orbit of the planet. It is this latter effect with which we are at present concerned.

I propose then to consider the probable relative importance of these two causes of change in the planetary orbits.

But before doing so it will be well as a preliminary to consider another point.

In considering the effects of tidal friction the theory has been throughout adopted that the tidally-disturbed body is homogeneous and viscous. Now we know that the planets are not homogeneous, and it seems not improbable that the tidally-disturbed parts will be principally more or less superficial—as indeed we know that they are in the case of terrestrial oceans. The question then arises as to the extent of error introduced by the hypothesis of homogeneity.

For a homogeneous viscous planet we have shown that the tidal frictional couple is approximately equal * to

$$C\frac{\tau^2}{\mathfrak{g}}\frac{n-\Omega}{\mathfrak{p}}, \text{ where } \mathfrak{p}=\frac{gaw}{19v}$$

Now how will this expression be modified, if the tidally-disturbed parts are more or less superficial, and of less than the mean density of the planet?

To answer this query we must refer back to the manner in which the expression was built up.

By reference to my paper "On the Tides of a Viscous Spheroid" (Phil. Trans., Part I., 1879, pp. 8–10, especially the middle of p. 8), it will be seen that $\mathfrak{p}$ is really $(\frac{5}{2}gaw-\frac{3}{2}gaw)/19v$, and that in both of these terms w represents the density of the tidally-disturbed matter, but that in the former g represents the gravitation of the planet and in the second it is equal to $\frac{4}{3}\pi\mu aw$, where w is the density of the tidally-disturbed matter. Now let f be the ratio of the mean density of the spheroid to the density of the tidally-disturbed matter.

Then in the former term

$$gaw=\frac{3}{4\pi\mu}\cdot g\cdot\tfrac{4}{3}\pi\mu aw=\frac{3}{4\pi\mu}g^2\times\frac{1}{f}$$

And in the latter

$$gaw=\frac{3}{4\pi\mu}g^2\times\frac{1}{f^2}$$

Hence if the planet be heterogeneous and the tidally-disturbed matter superficial, $\mathfrak{p}$ must be a coefficient of the form

$$\frac{3}{4\pi\mu\times 19}\frac{g^2}{vf}\left(\frac{5}{2}-\frac{3}{2f}\right)$$

If f be unity this reduces to the form $gaw/19v$, as it ought; but if the tidally-

* I leave out of account the case of "large" viscosity, because as shown in a previous paper that could only be true of a planet which in ordinary parlance would be called a solid of great rigidity.—See "Precession," Phil. Trans., 1879, Part II., p. 531.

disturbed matter be superficial and of less than the mean density, then $\mathfrak{p}$ must be a coefficient which varies as $\frac{g^2}{\mu v f}\left(1-\frac{3}{5f}\right)$. The exact form of the coefficient will of course depend upon the exact nature of the tides. If f be large the term $3/5f$ will be negligeable compared with unity. Again, if we refer to the following paper ("Precession, &c.," Phil. Trans., Part II., 1879, p. 456), it will be seen that the C in the expression for the tidal frictional couple represents $\frac{2}{5}(\frac{4}{3}\pi a^3 w)a^2$, where w is the density of the tidally-disturbed matter; hence C should be replaced by C/f.

Then if we reconstruct the expression for the tidal frictional couple, we see that it is to be divided by f, because of the true meaning to be assigned to C, but is to be multiplied by f on account of the true meaning to be assigned to $\mathfrak{p}$.

From this it follows that for a given viscosity it is, roughly speaking, probable that the tidal frictional couple will be nearly the same as though the planet were homogeneous. The above has been stated in an analytical form, but in physical language the reason is because the lagging of the tide will be augmented by the deficiency of density of the tidally-disturbed matter in about the same proportion as the frictional couple is diminished by the deficiency of density of the tide-wave upon which the disturbing satellite has to act.

This discussion appeared necessary in order to show that the tidal frictional couple is of the same order of magnitude whether the planet be homogeneous or heterogeneous, and that we shall not be led into grave errors by discussing the theory of tidal friction on the hypothesis of the homogeneity of the tidally-disturbed bodies.

We may now proceed to consider the double tidal action of a planet and the sun.

Let us consider the particular homogeneous planet whose mass, distance from sun, and orbital angular velocity are m, c, Ω. For this planet, let C' = moment of inertia; a' = mean radius; w' = density; g' = gravity; $\mathfrak{g}'=\frac{2}{5}g'/a'$; v' = the viscosity; $\mathfrak{p}'=g'a'w'/19v'=\frac{3}{4\times 19}g'^2/\pi\mu v'$; and n' = angular velocity of diurnal rotation.

The same symbols when unaccented are to represent the parallel quantities for the sun.

Now suppose the sun to be either perfectly rigid, perfectly elastic, or perfectly fluid. Then *mutatis mutandis*, equation (2) gives the rate of increase of the planet's distance from the sun under the influence of the tidal friction in the planet. It becomes

$$\mu^{\frac{1}{2}}\frac{Mm}{(M+m)^{\frac{1}{2}}}\frac{dc^{\frac{1}{2}}}{dt}=\left(\tfrac{3}{2}\right)^2\frac{C'}{\mathfrak{g}'}\frac{(\mu M)^2}{c^6}\frac{n'-\Omega}{\mathfrak{p}'}$$

If the planet have no satellite the right-hand side is equal to $-C'dn'/dt$, because the equation was formed from the expression for the tidal frictional couple.

Hence, if none of the planets had satellites we should have a series of equations of the form

$$C'n'+\mu^{\frac{1}{2}}\frac{Mm}{(M+m)^{\frac{1}{2}}}c^{\frac{1}{2}}=h$$

with different h's corresponding to each planet.

We may here remark that the secular effects of tidal friction in the case of a rigid sun attended by tidally-disturbed planets, with no satellites, may easily be determined. For if we put $c^{\frac{1}{2}}=x$, and note that Ω varies as x^{-3}, and that n' has the form $(h-kx)/C'$, we see that it would only be necessary to evaluate a series of integrals of the form $\int_{x_0}^{x}\frac{\epsilon x^{15}dx}{\alpha-\beta x^3+\gamma x^4}$. This integral is in fact merely the time which elapses whilst x changes from x_0 to x, and the time scale is the same for all the planets. It is not at present worth while to pursue this hypothetical case further.

Now if we suppose the planet to raise frictional tides in the sun, as well as the sun to raise tides in the planets, we easily see by a double application of (2) that

$$\mu^{\frac{1}{2}}\frac{Mm}{(M+m)^{\frac{1}{2}}}\frac{dc^{\frac{1}{2}}}{dt}=\left(\tfrac{3}{2}\right)^2\frac{1}{c^6}\left[(\mu m)^2\frac{C}{\mathfrak{g}\mathfrak{p}}(n-\Omega)+(\mu M)^2\frac{C'}{\mathfrak{g}'\mathfrak{p}'}(n'-\Omega)\right] \quad . \quad . \quad . \quad (28)$$

The tides raised in the planet by its satellites do not occur explicitly in this equation, but they do occur implicitly, because n', the planet's rotation, is affected by these tides.

The question which we now have to ask is whether in the equation (28) the solar term (without accents) or the planetary term (with accents) is the more important.

In the solar system the rotations of the sun and planets are rapid compared with the orbital motions, so that Ω may be neglected compared with both n and n'.

Hence the planetary term bears to the solar term approximately the ratio $\frac{M^2C'n'\mathfrak{g}\mathfrak{p}}{m^2Cn\mathfrak{g}'\mathfrak{p}'}$.

Now $\left(\frac{M}{m}\right)^2\frac{C'}{C}\frac{\mathfrak{g}}{\mathfrak{g}'}=\frac{M}{m}\left(\frac{a'}{a}\right)^2\frac{g}{g'}\frac{a'}{a}=\left(\frac{g}{g'}\right)^2\frac{a'}{a}$. Also $\frac{\mathfrak{p}}{\mathfrak{p}'}=\left(\frac{g}{g'}\right)^2\frac{v'}{v}$.

Therefore the ratio is $\left(\frac{g}{g'}\right)^4\frac{a'}{a}\frac{n'}{n}\frac{v'}{v}$

Now solar gravity is about 26·4 times that of the earth and about 10·4 times that of Jupiter. The solar radius is about 109 times that of the earth and about 10 times that of Jupiter. The earth's rotation is about 25·4 times that of the sun, and Jupiter's rotation is about 61 times that of the sun. Combining these data I find that the effect of solar tides in the earth is about 113,000 v'/v times as great as the effect of terrestrial tides in the sun, and the effect of solar tides in Jupiter is about 70,000 v'/v times as great as the effect of Jovian tides in the sun. It is not worth while to make a similar comparison for any of the other planets.

Now it seems reasonable to suppose that the coefficient of tidal friction in the planets is of the same order of magnitude as in the sun, so that it is improbable that v'/v should be either a large number or a small fraction.

We may conclude then from this comparison that the effects of tides raised in the sun by the planets are quite insignificant in comparison with those of tides raised in the planets by the sun.

It appears therefore that we may fairly leave out of account the tides raised in the sun in studying the possible changes in the planetary orbits as resulting from tidal friction.

But the difference of physical condition in the several planets is probably considerable, and this would lead to differences in the coefficients of tidal friction to which there is no apparent means of approximating. It therefore seems inexpedient at present to devote time to the numerical solution of the problem of the rigid sun and the tidally-disturbed planets.

§ 7. *Numerical data and deductions therefrom.*

Although we are thus brought to admit that it is difficult to construct any problem which shall adequately represent the actual case, yet a discussion of certain numerical values involved in the solar system and in the planetary subsystems will, I think, lead to some interesting results.

The fundamental fact with regard to the theory of tidal friction is the transformation of the rotational momentum of the planet as it is destroyed by tidal friction into orbital momentum of the tide-raising body.

Hence we may derive information concerning the effects of tidal friction by the evaluation of the various momenta of the several parts of the solar system.

Professor J. C. ADAMS has kindly given me a table of values of the planetary masses, each with its attendant satellites. The authorities were as follows: for Mercury, ENCKE; for Venus, LE VERRIER; for the Earth, HANSEN; for Mars, HALL; for Jupiter, BESSEL; for Saturn, BESSEL; for Uranus, VON ASTEN; for Neptune, NEWCOMB.

The masses were expressed as fractions of the sun. The results, when earth plus moon is taken as unity, are given in the table below. The mean distances, taken from HERSCHEL'S 'Astronomy,' are given in a second column.

	Masses (m.).	Mean Distances (c.).
Sun . . .	315,511·	
Mercury . .	·06484	·387098
Venus . .	·78829	·723332
Earth . .	1·00000	1·000000
Mars . . .	·10199	1·523692
Jupiter . .	301·0971	5·202776
Saturn . .	90·1048	9·538786
Uranus . .	14·3414	19·18239
Neptune. .	16·0158	30·05660

The unit of mass is earth plus moon, the unit of length is the earth's mean distance from the sun, and the unit of time will be taken as the mean solar day.

Then μ being the attraction between unit masses at unit distance, M being sun's mass, and 365·25 being the earth's periodic time, we have

$$\sqrt{\mu M}=\frac{2\pi}{365\cdot25}=\frac{10^{8\cdot23558}}{10^{10}}$$

The momentum of orbital motion of any one of the planets round the sun is given by $m.\sqrt{\mu M}.\sqrt{c}$.

With the above data I find the following results.*

TABLE I.

Planet.	Orbital momentum.
Mercury	·00079
Venus	·01309
Earth	·01720
Mars	·00253
Jupiter	13·469
Saturn	5·456
Uranus	1·323
Neptune	1·806
Total	22·088

We must now make an estimate of the rotational momentum of the sun, so as to compare it with the total orbital momentum of the planets.

It seems probable that the sun is much more dense in the central portion, than near the surface.† Now if the Laplacian law of internal density were to hold with the sun, but with the surface density infinitely small compared with the mean density, we should have

$$C=\tfrac{2}{3}\left[1-\frac{6}{\pi^2}\right]Ma^2$$

If on the other hand the sun were of uniform density we should have $C=\tfrac{2}{5}Ma^2$‡.

* These values are of course not rigorously accurate, because the attraction of Jupiter and Saturn on the internal planets is equivalent to a diminution of the sun's mass for them, and the attraction of the internal planets on the external ones is equivalent to an increase of the sun's mass.

† I have elsewhere shown that there is a strong probability that this is the case with Jupiter, and that planet probably resembles the sun more nearly than does the earth.—See Ast. Soc. Month. Not., Dec., 1876.

‡ These considerations lead me to remark that in previous papers, where the tidal theory was applied numerically to the case of the earth and moon, I might have chosen more satisfactory numerical values with which to begin the computations.

It was desirable to use a consistent theory of frictional tides, and that founded on the hypothesis of a homogeneous viscous planet was adopted.

The earth had therefore to be treated as homogeneous, and since tidal friction depends on relative

The former of these two suppositions seems more likely to be near the truth than the latter.

Now $\frac{2}{3}(1-6\pi^{-2})=\cdot 26138$, so that C may lie between $(\cdot 26138)Ma^2$ and $(\cdot 4)Ma^2$.

The sun's apparent radius is $961''\cdot 82$, therefore the unit of distance being the present distance of the earth from the sun, $a=961\cdot 82\pi/648{,}000$; also $M=315{,}511$.

Lastly the sun's period of rotation is about 25·38 m.s. days, so that $n=2\pi/25\cdot 38$.

Combining these numerical values I find that Cn (the solar rotational momentum) may lie between ·444 and ·679. The former of these values seems however likely to be far nearer the truth than the latter.

It follows therefore that the total orbital momentum of the planetary system, found above to be 22, is about 50 times that of the solar rotation.

motion, the rotation of the homogeneous planet had to be made identical with that of the real earth. A consequence of this is that the rotational momentum of the earth in my problem bore a larger ratio to the orbital momentum of the moon than is the case in reality. Since the consequence of tidal friction is to transfer momentum from one part of the system to the other, this treatment somewhat vitiated subsequent results, although not to such an extent as could make any important difference in a speculative investigation of that kind.

If it had occurred to me, however, it would have been just as easy to have replaced the actual heterogeneous earth by a homogeneous planet mechanically equivalent thereto. The mechanical equivalence referred to lies in the identity of mass, moment of inertia, and rotation between the homogeneous substitute and the real earth. These identities of course involve identity of rotational momentum and of rotational energy, and, as will be seen presently, other identities are approximately satisfied at the same time.

Suppose that roman letters apply to the real earth and italic letters to the homogeneous substitute.

By LAPLACE's theory of the earth's figure, with THOMSON and TAIT's notation ('Nat. Phil.,' § 824)

$$\mathrm{C}=\tfrac{2}{3}\left\{1-\frac{6(f-1)}{f\theta^2}\right\}\mathrm{Ma}^2$$

where f is the ratio of mean to surface density, and θ is a certain angle.

Also
$$C=\tfrac{2}{5}Ma^2\left(1+\frac{2e}{3}\right)$$

where e is the ellipticity of the homogeneous planet's figure.

Then by the above conditions of mechanical identity

$$\mathrm{M}=M \text{ and } \mathrm{C}=C$$

whence
$$\left(\frac{a}{\mathrm{a}}\right)^2=\tfrac{5}{3}\left(1-\frac{2e}{3}\right)\left\{1-\frac{6(f-1)}{f\theta^2}\right\}$$

Now put $\mathrm{m}=\mathrm{n}^2\mathrm{a}/\mathrm{g}$, $m=n^2a/g$; where g, g are mean pure gravity in the two cases. Then the remaining condition gives $\mathrm{n}=n$.

Therefore
$$\frac{m}{\mathrm{m}}=\frac{\mathrm{g}a}{g\mathrm{a}}=\left(\frac{a}{\mathrm{a}}\right)^3$$

But
$$e=\tfrac{5}{4}m=\tfrac{5}{4}\mathrm{m}\left(\frac{a}{\mathrm{a}}\right)^3$$

Hence
$$\left(\frac{a}{\mathrm{a}}\right)^2=\tfrac{5}{3}\left\{1-\tfrac{5}{6}\mathrm{m}\left(\frac{a}{\mathrm{a}}\right)^3\right\}\left\{1-\frac{6(f-1)}{f\theta^2}\right\}$$

This is an equation which gives the radius of the homogeneous substituted planet in terms of that of

In discussing the various planetary subsystems I take most of the numerical values from the excellent tables of astronomical constants in Professor BALL'S 'Astronomy,'* and from the table of masses given above.

Mercury.

The diameter at distance unity is about 6″·5; the diurnal period is $24^h\ 0^m\ 50^s$ (?) The value of the mass seems very uncertain, but I take ENCKE'S value given above. Assuming that the law of internal density is the same as in the earth (see below), we have $C = \cdot 33438ma^2$, and $n = 2\pi$ very nearly. Whence I find for the rotational momentum

$$Cn = \frac{\cdot 34}{10^{10}}.$$

Venus.

The diameter at distance unity is about 16″·9; the diurnal period is $23^h\ 21^m\ 22^s$ (?). Assuming the same law of internal density as for the earth, I find

$$Cn = \frac{28 \cdot 6}{10^{10}}.$$

HERSCHEL remarks ('Outlines of Astronomy,' § 509) that "both Mercury and Venus have been concluded to revolve on their axes in about the same time as the Earth, though in the case of Venus, BIANCHINI and other more recent observers have contended

the earth. It may be solved approximately by first neglecting $\frac{5}{6}\mathrm{m}(a/\mathrm{a})^3$, and afterwards using the approximate value of a/a for determining that quantity.

The density of the homogeneous planet is found from

$$w = \mathrm{w}\left(\frac{\mathrm{a}}{a}\right)^3$$

where w is the earth's mean density.

To apply these considerations to the earth, we take $\theta = 142^\circ\ 30'$, $f = 2 \cdot 057$, which give $\frac{1}{295}$ as the ellipticity of the earth's surface.

Then with these values (THOMSON and TAIT'S 'Nat. Phil.,' § 824, table, col. vii., they give however ·835)

$$\tfrac{5}{3}\left\{1 - \frac{6(f-1)}{f\theta^2}\right\} = \cdot 83595$$

The first approximation gives $\frac{a}{\mathrm{a}} = \cdot 9143$, and the second $\frac{a}{\mathrm{a}} = \cdot 9133$.

Hence the radius of the actual earth 6,370,000 meters becomes, in the homogeneous substitute, 5,817,000 meters.

Taking 5·67 as the earth's mean specific gravity, that of the homogeneous planet is 7·44.

The ellipticity of the homogeneous planet is ·00329 or $\frac{1}{303}$, which differs but little from that of the real earth, viz.: $\frac{1}{295}$.

The precessional constant of the homogeneous planet is equal to the ellipticity, and is therefore ·00329. If this be compared with the precessional constant ·00327 of the earth, we see that the homogeneous substitute has sensibly the same precession as has the earth.

If a similar treatment be applied to Jupiter, then (with the numerical values given in a previous paper, Ast. Soc. Month. Not., Dec. 1876) the homogeneous planet has a radius equal to ·8 of the actual one; its density is about half that of the earth, and its ellipticity is $\frac{1}{19}$.

* 'Text Book of Science: Elements of Astronomy.' LONGMANS. 1880.

for a period of twenty-four times that length." He evidently places little reliance on the observations.

The Earth.

I adopt LAPLACE'S theory of internal density (with THOMSON and TAIT'S notation), and take, according to Colonel CLARKE, the ellipticity of surface to be $\frac{1}{295}$. This value corresponds with the value 2·057 for the ratio of mean to surface density (the f of THOMSON and TAIT), and to 142° 30′ for the auxiliary angle θ.

The moment of inertia is given by the formula

$$C=\tfrac{2}{3}\left[1-\frac{6(f-1)}{f\theta^2}\right]ma^2.$$

These values of θ and f give $\frac{5}{3}\left[1-\frac{6(f-1)}{f\theta^2}\right]=\cdot 83595.$

Whence $$C=\cdot 33438ma^2.$$

The numerical coefficient is the same as that already used in the case of the two previous planets.

The moon's mass being $\frac{1}{82}$nd of the earth's, the earth's mass is $\frac{82}{83}$ in the chosen unit of mass.

With sun's parallax 8″·8, and unit of length equal to earth's mean distance

$$a=\frac{8\cdot 8\pi}{648000}$$

The angular velocity of diurnal rotation, with unit of time equal to the mean solar day,

$$n=\frac{2\pi}{\cdot 99727}$$

Combining these values I find for the earth's rotational momentum

$$Cn=\frac{37\cdot 88}{10^{10}}$$

Writing m' for the moon's mass, and neglecting the eccentricity of the lunar orbit, the moon's orbital momentum* is

$$\frac{mm'}{m+m'}\Omega c^2$$

* If we determine μ from the formula

$$\mu=\Omega^2c^3=\left(\frac{2\pi}{27\cdot 3217}\right)^2\left(\frac{8\cdot 8}{3422\cdot 3}\right)^3$$

and observe that $\mu M=(2\pi/365\cdot 25)^2$, we obtain 329,000 as the sun's mass. This disagrees with the value 315,511 used elsewhere. The discrepancy arises from the neglect of solar perturbation of the moon, and of planetary perturbation of the earth.

Taking the moon's parallax as $3422''\cdot3$ (which gives a distance of 60·27 earth's radii), and the sun's parallax as $8''\cdot8$, we have

$$c=\frac{8\cdot8}{3422\cdot3}$$

Taking the lunar period as 27·3217 m.s. days we have

$$\Omega=\frac{2\pi}{27\cdot3217}$$

As above stated, m is $\frac{82}{83}$, and m' is $\frac{1}{83}$; whence it will be found that the moon's orbital momentum is $\frac{181}{10^{10}}$.

This is 4·78 times the earth's rotational momentum.

The resultant angular momentum of the system, with obliquity of ecliptic 23° 28′, is 5·71 times the earth's rotational momentum, and is $\frac{216}{10^{10}}$.

Mars.

The polar diameter at distance unity is $9''\cdot352$ (HARTWIG, 'Nature,' June 3, 1880). With an ellipticity $\frac{1}{300}$ this gives $4''\cdot686$ as the mean radius. The diurnal period is $24^h\ 37^m\ 23^s$. Assuming the law of internal density to be the same as in the earth I find

$$Cn=\frac{1\cdot08}{10^{10}}$$

The masses of the satellites are very small, and their orbital momentum must also be very small.

Jupiter.

The polar and equatorial diameters at the planet's mean distance from the sun are $35''\cdot170$ and $37''\cdot563$ (KAISER and BESSEL, 'Ast. Nach,' vol. 48, p. 111). These values give a mean radius $5\cdot2028\times18''\cdot383$ at distance unity.

The period of rotation is $9^h\ 55\frac{1}{2}^m$, or ·4136 m.s. day.

I have elsewhere shown reason to believe that the surface density of Jupiter is very small compared with the mean density. It appears that we have approximately

$$C=\tfrac{2}{3}\times\frac{1}{2\cdot528}ma^2=\cdot2637\ ma^2.*$$

The numerical coefficient differs but little from that which we should have, if the Laplacian law of internal density were true, with infinitely small surface density (f infinite, $\theta=180°$); for, as appeared in considering the sun's moment of inertia, the factor would be in that case ·26138.

* Ast. Soc. Month. Not., Dec., 1876, p. 83.

With these values I find

$$Cn=\frac{2,594,000}{10^{10}}=\cdot 0002594.$$

The distances of the satellites referred to the mean distance of Jupiter from the sun are

I.	II.	III.	IV.
111″·74	177″·80	283″·61	498″·87

Then taking Jupiter's mean distance to be 5·20278, the logarithms of the distances in terms of the earth's distance from the sun are

I.	II.	III.	IV.
7·45002—10	7·65174—10	7·85453—10	8·09980—10

The periodic times are in m.s. days (HERSCHEL'S 'Astronomy,' Appendix)

I.	II.	III.	IV.
1·76914	3·55181	7·15455	16·6888

The masses given me by Professor ADAMS* from a revision of DAMOISEAU'S work are in terms of Jupiter's mass.

I.	II.	III.	IV.
$\frac{2\cdot 8311}{10^5}$	$\frac{2\cdot 3236}{10^5}$	$\frac{8\cdot 1245}{10^5}$	$\frac{2\cdot 1488}{10^5}$

Combining these data according to the formula $m\Omega c^2$, where m is the mass of the satellite, I find for the orbital momenta of the satellites expressed in terms of the chosen units—

I.	II.	III.	IV.
$\frac{2406}{10^{10}}$	$\frac{2489}{10^{10}}$	$\frac{10993}{10^{10}}$	$\frac{3857}{10^{10}}$

The sum of these is $19745/10^{10}$ and is the total orbital momentum of the satellites. It is $\frac{1}{130}$th of the rotational momentum of the planet as found above.

The whole angular momentum of the Jovian system is $\frac{2,614,000}{10^{10}}$.

Saturn.

There seems to be much doubt as to the diameter of the planet.

The values of the *mean* radius at distance unity given by BESSEL, DE LA RUE, and

* He kindly gave me these data for another purpose.—See Ast. Soc. Month. Not., Dec., 1876, p. 81.

MAIN (with ellipticities 1/10·2, 1/11(?), 1/9·227 respectively) are 79″, 82″, and 94″ respectively.*

The period of rotation is $10^h\ 29\frac{1}{4}^m$ or ·437 m.s. day.

Assuming (as with the sun) that the surface density is infinitely small compared with the mean density, we have $C = \cdot 2614 ma^2$. I find then that these three values give respectively,

$$Cn = \frac{497{,}000}{10^{10}} \text{ and } \frac{535{,}000}{10^{10}} \text{ and } \frac{703{,}000}{10^{10}}$$

The masses of the satellites are unknown, but HERSCHEL thinks that Titan is nearly as large as Mercury.

If we take its mass as ·06 in terms of the earth's mass, its distance as 176″·755 at the planet's mean distance from the sun, and its periodic time as 15·95 m.s. days, we find the orbital momentum to be $16{,}000/10^{10}$. The whole orbital momentum of the satellites and the ring is likely to be greater than this, for the ring has been variously estimated to have a mass equal to $\frac{1}{120}$th to $\frac{1}{620}$th of the planet.

It is probable therefore that orbital momentum of the system is $\frac{1}{30}$th, or thereabouts, of the rotational momentum of the planet.

Nothing is known concerning the rotation of Uranus and Neptune, and but little of their satellites.

The results of this numerical survey of the planets are collected in the following table.

TABLE II.

Planet.	i. Rotational momentum of planet $\times 10^{10}$.	ii. Orbital momentum of satellites $\times 10^{10}$.	iii. Ratio of ii. to i.	iv. Total momentum of each planet's system $\times 10^{10}$.
Mercury . . .	·34 ?	. .	. .	·34 ?
Venus . . .	28·6 ?	. .	. .	28·6 ?
Earth. . . .	37·88	181	4·78	216
Mars	1·08	very small	very small	1·08
Jupiter . . .	2,594,000	20,000	$\frac{1}{130}$	2,614,000
Saturn . . .	500,000 to 700,000	16,000 or more	$\frac{1}{30}$ or more	520,000 to 720,000

The numbers marked with queries are open to much doubt.

If the numbers given in column iv. of this table be compared with those given in Table I., it will be seen that the total internal momentum of each of the planetary

* Deduced from values of the equatorial diameter found by these observers, referred to the planet's mean distance from the sun, as given by BALL.

subsystems is very small compared with the orbital momentum of the planet in its motion round the sun. This ratio is largest in the case of Jupiter, and here the internal momentum is ·00026 whilst the orbital momentum is 13; hence in the case of Jupiter the orbital momentum is 5000 times the sum of the rotational momentum of the planet and the orbital momentum of its satellites. From this it follows that if the whole of the momentum of Jupiter and his satellites were destroyed by solar tidal friction, the mean distance of Jupiter from the sun would only be increased by $\frac{1}{2500}$th part. The effect of the destruction of the internal momentum of any of the other planets would be very much less.

If therefore the orbits of the planets round the sun have been considerably enlarged, during the evolution of the system, by the friction of the tides raised in the planets by the sun, the primitive rotational momentum of the planetary bodies must have been thousands of times greater than at present. If this were the case then the enlargement of the orbits must simultaneously have been somewhat increased by the reaction of the tides raised in the sun by the planets.

But it does not seem probable that the planetary masses ever possessed such an enormous amount of rotational momentum, and therefore it is not probable that tidal friction has considerably affected the dimensions of the planetary orbits.

It is difficult to estimate the degree of attention which should be paid to BODE's empirical law concerning the mean distances of the planets, but it may perhaps be supposed that that law (although violated in the case of Neptune, and only partially satisfied by the asteroids) is the outcome of the laws governing the successive epochs of instability in the history of a rotating and contracting nebula. Now if, after the genesis of the planets, tidal friction had considerably affected the planetary distances, then all appearance of such primitive law in the distances would be thereby obliterated. If therefore there be now observable a sort of law of mean distances, it to some extent falls in with the conclusion arrived at by the preceding numerical comparisons.

The extreme relative smallness of the masses of the Martian and Jovian satellites tends to show the improbability of very large changes in the dimensions of the orbits of those satellites; although the argument has not nearly equal force in these cases, because the distances of the satellites from these planets is small.

The numbers given in column iii. of Table II. show in a striking manner the great difference between the present physical conditions of the terrestrial system and those of Mars, Jupiter, and Saturn. These numbers may perhaps be taken as representing the amount of effect which the tidal friction due to the satellites has had in their evolution, and confirms the conclusion that, whilst tidal friction may have been (and according to previous investigations certainly appears to have been) the great factor in the evolution of the earth and moon, yet with the satellites of the other planets it has not had such important effects.

In previous papers the expansion of the lunar orbit under the influence of terrestrial tidal friction was examined, and the moon was traced back to an origin close to the present surface of the earth. The preceding numerical comparisons suggest that the contraction of the planetary masses has elsewhere been the more important factor, and that the genesis of satellites occurred elsewhere earlier in the evolution.

It has been shown that the case of the earth and moon does actually differ widely from that of the other planets, and we may therefore reasonably suppose that the history has also differed considerably.

Although we might perhaps leave the subject at this point, yet, after arriving at the above conclusions, it seems natural to inquire in what manner the simultaneous action of the contraction of a planetary mass and of tidal friction is likely to have operated.

The subject is necessarily speculative, but the conclusions at which I arrive are, I think, worthy of notice, for although they involve much of mere conjectural assumption in respect to the quantities and amounts assumed, yet they are deduced from the rigorous dynamical principles of angular momentum and of energy.

§ 8. *On the part played by tidal friction in the evolution of planetary masses.*

To consider the subject of this section, we require--

(α) Some measure of the relative efficiency of solar tidal friction in reducing the rotational momentum and the rotation of the several planets.

(β) We have to consider the manner in which the simultaneous action of the contraction of the planetary mass and of solar tidal friction co-operate.

(γ) We have to discuss how the separation of a satellite from the contracting mass is likely to affect the course of evolution.

It is not possible to treat these questions rigorously, but without some guidance on these points further discussion would be fruitless.

The probable influence of the heterogeneity of the planetary mass on tidal friction has been already discussed, and it has been shown that the case of homogeneity will probably give good indications of the result in the true case. I therefore adhere here also to the hypothesis of homogeneity.

I will begin with (α) and consider—

The relative efficiency of solar tidal friction.

The rate at which the rotation of any one of the planets is being reduced is $\tau^2(n-\Omega)/\mathfrak{g}\mathfrak{p}$, where n, $\mathfrak{g}$, $\mathfrak{p}$ refer to the planet, and are the quantities which were previously indicated by the same symbols accented.

τ is $\frac{3}{2}M/c^3$, and therefore varies as Ω^2. With all the planets (excepting, perhaps, Mercury and Venus, according to HERSCHEL) Ω is small compared with n, and we may write n for $n-\Omega$.

It has been already shown that $\mathfrak{p}=\frac{3}{19\times 4\pi}\frac{g^2}{\mu v}$, and $\mathfrak{g}=\frac{2}{5}\frac{g}{a}$.

Hence
$$\mathfrak{p}\mathfrak{g}=\frac{2\times 3}{5\times 19\times 4\pi}\frac{g^3}{\mu a v}=\frac{2\times 3}{5\times 19\times 4\pi}\frac{\mu^2 m^3}{a^7 v}$$

Therefore the rate of reduction of planetary rotation is proportional to $\Omega^4 a^7 n m^{-3} v$.

The coefficient of friction v is quite unknown, but we shall obtain indications of the relative importance of tidal retardation in the several planets by supposing v to be the same in all. If we multiply this expression by ma^2, we obtain an expression to which the rate of reduction of rotational momentum is proportional. By means of the data used in the preceding section I find the following results.

TABLE III.

Planet.	Number to which tidal retardation is proportional.	Number to which rate of destruction of rotational momentum is proportional.
Mercury	1000· (?)	9·1 (?)
Venus	11· (?)	8·1 (?)
Earth	1·	1·0
Mars	·89	·026
Jupiter	·00005	2·3
Saturn	·000020 to ·000066	·11 to ·54

This table only refers to solar tidal friction, and the numbers are computed on the hypothesis of the identity of the coefficient of tidal friction for all the planets.

The figures attached to Mercury and Venus are open to much doubt. Perhaps the most interesting point in this table is that the rate of solar tidal retardation of Mars is nearly equal to that of the earth, notwithstanding the comparative closeness of the latter to the sun. The significance of these figures will be commented on below.

I shall now consider—

(β) *The manner in which solar tidal friction and the contraction of the planetary nebula work together.*

It will be supposed that the contraction is the more important feature, so that the acceleration of rotation due to contraction is greater than the retardation due to tidal friction.

Let h be the rotational momentum of the planet at any time; then

$$Cn=h \text{ or } n=\frac{5}{2}\frac{h}{ma^2} \quad . \quad . \quad . \quad . \quad (29)$$

In accordance with the above supposition h is a quantity which diminishes slowly in consequence of tidal friction, and a diminishes in consequence of contraction, at such a rate that dn/dt is positive.

We also have
$$\mathfrak{pg}=\frac{2\times 3}{19\times 5\times 4\pi}\,\frac{g^3}{\mu v a}.$$

The rate of change in the dimensions of the planet's orbit about the sun remains insensible, so that τ and Ω may be treated as constant.

Then the rate of loss of rotational momentum of the planetary mass is $Cn\frac{\tau^2}{\mathfrak{gp}}\left(1-\frac{\Omega}{n}\right)$.

By the above transformations we see that this expression varies as $\frac{hva}{g^3}\left(1-\frac{2}{5}\frac{\Omega m a^2}{h}\right)$.

But $m=\frac{4}{3}\pi w a^3$, and therefore $a=\left(\frac{3m}{4\pi}\right)^{\frac{1}{3}}w^{-\frac{1}{3}}$. Also $g=\mu m/a^2$.

On substituting this expression for g, and then replacing a throughout by its expression in terms of w, we see that, on omitting constant factors, the rate of loss of rotational momentum varies as $\frac{hv}{w^{\frac{7}{3}}}-\frac{kv}{w^3}$, where $k=\frac{2}{5}\left(\frac{3}{4\pi}\right)^{\frac{2}{3}}\Omega m^{\frac{5}{3}}$, a constant.

From (29) we see that if h varies as a^2, or as $w^{-\frac{2}{3}}$, n the angular velocity of rotation remains constant.

If therefore we suppose h to vary as some power of a less than 2 (which power may vary from time to time) we represent the hypothesis that the contraction causes more acceleration of rotation than tidal friction causes retardation. Let us suppose then that $h=\mathrm{H}w^{-\frac{2}{3}+\beta}$ where β is less than $\frac{2}{3}$, and varies from time to time.

Then the rate of loss of momentum varies as
$$vw^{-3}(\mathrm{H}w^{\beta}-k).$$

In order to determine the rate of loss of rotation we must divide this expression by C, which varies as $w^{-\frac{2}{3}}$.

Therefore the rate of loss of angular velocity of rotation varies as
$$vw^{-\frac{7}{3}}(\mathrm{H}w^{\beta}-k).$$

In order to determine how tidal friction and contraction co-operate it is necessary to adopt some hypothesis concerning the coefficient of friction v.

So long as the tides consist of a bodily distortion, the coefficient of friction must be some function of the density, and will certainly increase as the density increases.

Now if, as regards the loss of momentum, v varies as a power less than the cube of the density, the first factor vw^{-3} diminishes as the density increases; and if, as regards the loss of rotation, v varies as a power less than $\frac{7}{3}$ of the density, the first factor $vw^{-\frac{7}{3}}$ diminishes as the density increases.

And as the cube and $\frac{7}{3}$ powers both represent very rapid increments of the coefficient of friction with increase of density, it is probable that the first factor in both expressions diminishes as the contraction of the planetary mass proceeds.

Now consider the second factor $\mathrm{H}w^{\beta}-k$, which corresponds to the factor $n-\Omega$ in the expression for the rate of loss of rotational momentum ; Hw^6 is large compared with k so long as the rotation of the planet is fast compared with the orbital motion of the planet about the sun, and since this factor is always positive, it always increases as the contraction increases.

For planets remote from the sun, where contraction has played by far the more important part, β will be very nearly equal to $\frac{2}{3}$, and for those nearer to the sun β will be small (or it might be negative if the tidal retardation exceeds the contractional acceleration).

We thus have one factor always increasing and the other always diminishing, and the importance of the increasing factor is greater for planets remote from than for those near to the sun.

If β be small it is difficult to say how the two rates will vary as the contraction proceeds. But if β does not differ very much from $\frac{2}{3}$ both rates are probably initially small, then rise to a maximum and then diminish.

Hence it may be concluded as probable that in the history of a contracting planetary mass, which is sufficiently far from the sun to allow contraction to be a more important factor than tidal friction, both the rate of loss of rotational momentum and of loss of rotation, due to solar tidal friction, were initially small, rose to a maximum and then diminished.

These considerations are important as showing that the efficiency of solar tidal friction was probably greater in the past than at present.

We now come to (γ)—

The effect of the genesis of a satellite on the evolution.

This subject is necessarily in part obscure, and the conclusions must in so far be open to doubt.

When a satellite separates from a planetary mass, it seems probable that that part of the planetary mass, which before the change had the greatest angular momentum, is lost by the planet. Hence the rotational momentum of the planet suffers a diminution, and the mass is also diminished. An inspection of the expressions in the last paragraphs shows that it is probable that the loss of a satellite diminishes the rate of loss of planetary rotational momentum, but slightly increases the rate of loss of rotation due to solar tidal friction.

Now if the satellite be large the effect of the tides raised by the satellite in the planet is to cause a much more powerful reduction of planetary rotation than was effected by the sun. The rotational momentum thus removed from the planet reappears in the orbital momentum of its satellite. And the reduction of rotation of the planet causes a reduction of rate of solar tidal effects, by diminishing the angular velocity of the planet's rotation relatively to the sun.

The first and immediate effect of the separation of a satellite is no doubt highly speculative, but the second effect seems to follow undoubtedly, whatever be the mode of separation of the satellite.

From these considerations we may conclude that the effect of the separation of a satellite is to destroy planetary rotation, but to preserve angular momentum within the planetary subsystem.

Hence we ought to find that those planets which have large satellites have a slow rotation, but have a relatively large amount of angular momentum within their systems.

A proper method of comparison between the several planets is difficult of attainment, but these ideas seem to agree with the fact that the earth, which is large compared with Mars, rotates in the same time, but that the whole angular momentum of earth and moon is large.*

* A method of comparing the various members of the solar system has occurred to me, but it is not founded on rigorous argument.

It seems probable that the small density of the larger planets is due to their not being so far advanced in their evolution as the smaller ones, and it is likely that they are continuing to contract and will some day be as dense as the earth.

The proposed method of comparison is to estimate how fast each of the planets must rotate if, with their actual rotational momenta, they were as condensed as the earth, and had the same law of internal density.

The period of this rotation may be called the "effective period."

With the data used above, taking the earth's mean density as unity, the mean density of Mars is ·675, that of Jupiter ·235, that of Saturn ·125 or ·111 or ·074, according to the data used.

To condense these planets we must reduce their radii in the proportion of the cube-roots of these numbers.

Their actual moments of inertia must be reduced by multiplying by the $\frac{2}{3}$rd power of these numbers, and as we suppose the law of internal density to be the same as in the earth, the moments of inertia of Jupiter and Saturn must be also increased in the proportion ·33438 to ·26138.

Then the "effective period" will be the actual period reduced by the same factors as have been given for reducing the moments of inertia.

In this way I find that the Martian day is to be divided by 1·3; the Jovian day by 2; and the Saturnian day by 3·14 to 4·44 according to the data adopted. The earth's day of course remains unchanged.

The following table gives the results.

TABLE IV.

Planet.	Actual period of rotation.	Effective period of rotation.
Earth	$23^h\ 56^m$	$23^h\ 56^m$
Mars	$24^h\ 37^m$	19^h
Jupiter	$9^h\ 55^m$	5^h
Saturn	$10^h\ 29^m$	$3^h\ 20^m$ to $2^h\ 20^m$

This seems to me to illustrate the arguments used above. For there should in general be a diminution of effective period as we recede from the sun.

It will be noted that the earth, although ten times larger than Mars, has a longer effective period. The larger masses should proceed in their evolution slower than the smaller ones, and therefore the greater proximity of the earth to the sun does not seem sufficient to account for this, more especially as it is

§ 9. *General discussion and summary.*

According to the nebular hypothesis the planets and the satellites are portions detached from contracting nebulous masses. In the following discussion I shall accept that hypothesis in its main outline, and shall examine what modifications are necessitated by the influence of tidal friction.

In § 7 it is shown that the reaction of the tides raised in the sun by the planets must have had a very small influence in changing the dimensions of the planetary orbits round the sun, compared with the influence of the tides raised in the planets by the sun.

From a consideration of numerical data with regard to the solar system and the planetary subsystems, it appears improbable that the planetary orbits have been much enlarged by tidal friction, since the origin of the several planets. But it is possible that part of the eccentricities of the planetary orbits is due to this cause.

We must therefore examine the several planetary subsystems for the effects of tidal friction.

From arguments similar to those advanced with regard to the solar system as a whole, it appears unlikely that the satellites of Mars, Jupiter, and Saturn originated very much nearer the present surfaces of the planets than we now observe them. But the data being insufficient, we cannot feel sure that the alteration in the dimensions of the orbits of these satellites has not been considerable. It remains, however, nearly certain that they cannot have first originated almost in contact with the present surfaces of the planets, in the same way as, in previous papers, has been shown to be probable with regard to the moon and earth.

The numerical data in Table II., § 7, exhibit so striking a difference between the terrestrial system and those of the other planets, that, even apart from the considerations adduced in this and previous papers, we should have grounds for believing that the modes of evolution have been considerably different.

This series of investigations shows that the difference lies in the genesis of the moon close to the present surface of the planet, and we shall see below that solar tidal

shown above that the efficiency of solar tidal friction is of about the same magnitude for the two planets. It is explicable however by the considerations in the text, for it was there shown that a large satellite was destructive of planetary rotation.

If we estimate how fast the earth must rotate in order that the whole internal momentum of moon and earth should exist in the form of rotational momentum, then we find an effective period for the earth of $4^h\ 12^m$. This again illustrates what was stated above, viz.: that a large satellite is preservative of the internal momentum of the planet's system.

The orbital momentum of the satellites of the other planets is so small, that an effective period for the other planets, analogous to the $4^h\ 12^m$ of the earth, would scarcely differ sensibly from the periods given in the table.

If Jupiter and Saturn will ultimately be as condensed as the earth, then it must be admitted as possible or even probable that Saturn (and perhaps Jupiter) will at some future time shed another satellite; for the efficiency of solar tidal friction at the distance of Saturn is small, and a period of two or three hours gives a very rapid rotation.

friction may be assigned as a reason to explain how it happened that the terrestrial planet had contracted to nearly its present dimensions before the genesis of a satellite, but that this was not the case with the exterior planets.

The numbers given in Table III., § 8, show that the efficiency of solar tidal friction is very much greater in its action on the nearer planets than on the further ones. But the total amount of rotation of the various planetary masses destroyed from the beginning cannot be at all nearly proportional to the numbers given in that table, for the more remote planets must be much older than the nearer ones, and the time occupied by the contraction of the solar nebula from the dimensions of the orbit of Saturn down to those of the orbit of Mercury must be very long. Hence the time during which solar tidal friction has been operating on the external planets must be very much longer than the period of its efficiency for the interior ones, and a series of numbers proportional to the total amount of rotation destroyed in the several planets would present a far less rapid decrease, as we recede from the sun, than do the numbers given in Table III. Nevertheless the disproportion between these numbers is so great that it must be admitted that the effect produced by solar tidal friction on Jupiter and Saturn has not been nearly so great as on the interior planets.

In § 8 it has been shown to be probable that, as a planetary mass contracts, the rate of tidal retardation of rotation, and of destruction of rotational momentum increases, rises to a maximum, and then diminishes. This at least is so, when the acceleration of rotation due to contraction exceeds the retardation due to tidal friction; and this must in general have been the case. Thus we may suppose that the rate at which solar tidal friction has retarded the planetary rotations in past ages was greater than the present rate of retardation, and indeed there seems no reason why many times the present rotational momenta of the planets should not have been destroyed by solar tidal friction. But it remains very improbable that so large an amount of momentum should have been destroyed as to materially affect the orbits of the planets round the sun.

I will now proceed to examine how the differences of distance from the sun would be likely to affect the histories of the several planetary masses.

According to the nebular hypothesis a planetary nebula contracts, and rotates quicker as it contracts. The rapidity of the revolution causes its form to become unstable, or, perhaps a portion gradually detaches itself; it is immaterial which of these two really takes place. In either case the separation of that part of the mass, which before the change had the greatest angular momentum, permits the central portion to resume a planetary shape. The contraction and increase of rotation proceed continually until another portion is detached, and so on. There thus recur at intervals a series of epochs of instability or of abnormal change.

Now tidal friction must diminish the rate of increase of rotation due to contraction, and therefore if tidal friction and contraction are at work together, the epochs of instability must recur more rarely than if contraction acted alone.

If the tidal retardation is sufficiently great, the increase of rotation due to contraction will be so far counteracted as never to permit an epoch of instability to occur.

Now the rate of solar tidal frictional retardation decreases rapidly as we recede from the sun, and therefore these considerations accord with what we observe in the solar system.

For Mercury and Venus have no satellites, and there is a progressive increase in the number of satellites as we recede from the sun. Moreover, the number of satellites is not directly connected with the mass of the planet, for Venus has nearly the same mass as the earth and has no satellite, and the earth has relatively by far the largest satellite of the whole system. Whether this be the true cause of the observed distribution of satellites amongst the planets or not, it is remarkable that the same cause also affords an explanation, as I shall now show, of that difference between the earth with the moon, and the other planets with their satellites, which has caused tidal friction to be the principal agent of change with the former but not with the latter.

In the case of the contracting terrestrial mass we may suppose that there was for a long time nearly a balance between the retardation due to solar tidal friction and the acceleration due to contraction, and that it was not until the planetary mass had contracted to nearly its present dimensions that an epoch of instability could occur.

It may also be noted that if there be two equal planetary masses which generate satellites, but under very different conditions as to the degree of condensation of the masses, then the two satellites so generated would be likely to differ in mass; we cannot of course tell which of the two planets would generate the larger satellite. Thus if the genesis of the moon was deferred until a late epoch in the history of the terrestrial mass, the mass of the moon relatively to the earth, would be likely to differ from the mass of other satellites relatively to their planets.

If the contraction of the planetary mass be almost completed before the genesis of the satellite, tidal friction, due jointly to the satellite and to the sun, will thereafter be the great cause of change in the system, and thus the hypothesis that it is the sole cause of change will give an approximately accurate explanation of the motion of the planet and satellite at any subsequent time.

That this condition is fulfilled in the case of the earth and moon, I have endeavoured to show in the previous papers of this series.

At the end of the last of those papers the systems of the several planets were reviewed from the point of view of the present theory. It will be well to recapitulate shortly what was there stated and to add a few remarks on the modifications and additions introduced by the present investigation.

The previous papers were principally directed to the case of the earth and moon, and it was there found that the primitive condition of those bodies was as follows:—the earth was rotating, with a period of from two to four hours, about an axis inclined at 11° or 12° to the normal to the ecliptic, and the moon was revolving, nearly in contact with the earth, in a circular orbit coincident with the earth's equator, and with a periodic time only slightly exceeding that of the earth's rotation.

Then it was proved that lunar and solar tidal friction would reduce the system from this primitive condition down to the state which now exists by causing a retardation of terrestrial rotation, an increase of lunar period, an increase of obliquity of ecliptic, an increase of eccentricity of lunar orbit, and a modification in the plane of the lunar orbit too complex to admit of being stated shortly.

It was also found that the friction of the tides raised by the earth in the moon would explain the present motion of the moon about her axis, both as regards the identity of the axial and orbital revolutions, and as regards the direction of her polar axis.

Thus the theory that tidal friction has been the ruling power in the evolution of the earth and moon completely coordinates the present motions of the two bodies, and leads us back to an initial state when the moon first had a separate existence as a satellite.

This initial configuration of the two bodies is such that we are almost compelled to believe that the moon is a portion of the primitive earth detached by rapid rotation or other causes.

There may be some reason to suppose that the earliest form in which the moon had a separate existence was in the shape of a ring, but this annular condition precedes the condition to which the dynamical investigation leads back.

The present investigation shows, in confirmation of preceding ones,* that at this origin of the moon the earth had a period of revolution about the sun shorter than at present by perhaps only a minute or two, and it also shows that since the terrestrial planet itself first had a separate existence the length of the year can have increased but very little—almost certainly by not so much as an hour, and probably by not more than five minutes.†

With regard to the 11° or 12° of obliquity which still remains when the moon and earth are in their primitive condition, it may undoubtedly be partly explained by the friction of the solar tides before the origin of the moon, and perhaps partly also by the simultaneous action of the ordinary precession and the contraction and change of ellipticity of the nebulous mass.‡

* "Precession," § 19.

† If the change has been as much as an hour the rotational momentum of the earth destroyed by solar tidal friction must have been 33 times the present total internal momentum of moon and earth. For the orbital momentum of a planet varies as the cube root of its periodic time, and if we differentiate logarithmically we obtain the increment of periodic time in terms of the increment of orbital momentum. Then taking the numerical data from Tables I. and II. we see that this statement is proved by the fact that 3×33 times $[216 \div \cdot 01720 \times 10^{10}] \times 365 \cdot 25 \times 24$ is very nearly equal to unity.

‡ See a paper "On a Suggested Explanation of the Obliquity of Planets to their Orbits," 'Phil. Mag.,' March, 1877. See however § 21 "Precession."

In the review referred to I examined the eccentricities and inclinations of the orbits of the several other satellites, and found them to present indications favourable to the theory. In the present paper I have given reasons for supposing that the tidal friction arising from the action of the other satellites on their planets cannot have had so much effect as in the case of the earth. That those indications were not more marked, and yet seemed to exist, agrees well with this last conclusion.

The various obliquities of the planets' equators to their orbits were also considered, and I was led to conclude that the axes of the planets from Jupiter inwards were primitively much more nearly perpendicular to their orbits than at present. But the case of Saturn and still more that of Uranus (as inferred from its satellites) seem to indicate that there was a primitive obliquity at the time of the genesis of the planets, arising from causes other than those here considered.

The satellites of the larger planets revolve with short periodic times; this admits of a simple explanation, for the smallness of the masses of these satellites would have prevented tidal friction from being a very efficient cause of change in the dimensions of their orbits, and the largeness of the planets' masses would have caused them to proceed slowly in their evolution.

If the planets be formed from chains of meteorites or of nebulous matter the rotation of the planets has arisen from the excess of orbital momentum of the exterior over that of the interior matter. As we have no means of knowing how broad the chain may have been in any case, nor how much it may have closed in on the sun in course of concentration, we have no means of computing the primitive angular momentum of a planet. A rigorous method of comparison of the primitive rotations of the several planets is thus wanting.

If however the planets were formed under similar conditions, then, according to the present theory, we should expect to find the exterior planets now rotating more rapidly than the interior ones. It has been shown above (see Table IV., note to § 8) that, on making allowance for the different degrees of concentration of the planets, this is the case.

That the interior satellite of Mars revolves with a period of less than a third of its planet's rotation is perhaps the most remarkable fact in the solar system. The theory of tidal friction explains this perfectly,* and we find that this will be the ultimate

* It is proper to remark that the rapid revolution of this satellite might perhaps be referred to another cause, although the explanation appears very inadequate.

It has been pointed out above that the formation of a satellite out of a chain or ring of matter must be accompanied by a diminution of periodic time and of distance. Thus a satellite might after formation have a shorter periodic time than its planet.

If this, however, were the explanation, we should expect to find other instances elsewhere, but the case of the Martian satellite stands quite alone.

[I believe that I now (July, 1881) see some reason to suppose that the earliest form of a satellite may not be annular. The investigation necessary to test this idea seems likely to prove a difficult one.]

fate of all satellites, because the solar tidal friction retards the planetary rotation without directly affecting the satellite's orbital motion.

The numerical comparison in Table III. shows that the efficiency of solar tidal friction in retarding the terrestrial and Martian rotations is of about the same degree of importance, notwithstanding the much greater distance of the planet Mars.

From the discussion in this paper it will have been apparent that the earth and moon do actually differ from the other planets in such a way as to permit tidal friction to have been the most important factor in their history.

By an examination of the probable effects of solar tidal friction on a contracting planetary mass, we have been led to assign a cause for the observed distribution of satellites in the solar system, and this again has itself afforded an explanation of how it happened that the moon so originated that the tidal friction of the lunar tides in the earth should have been able to exercise so large an influence.

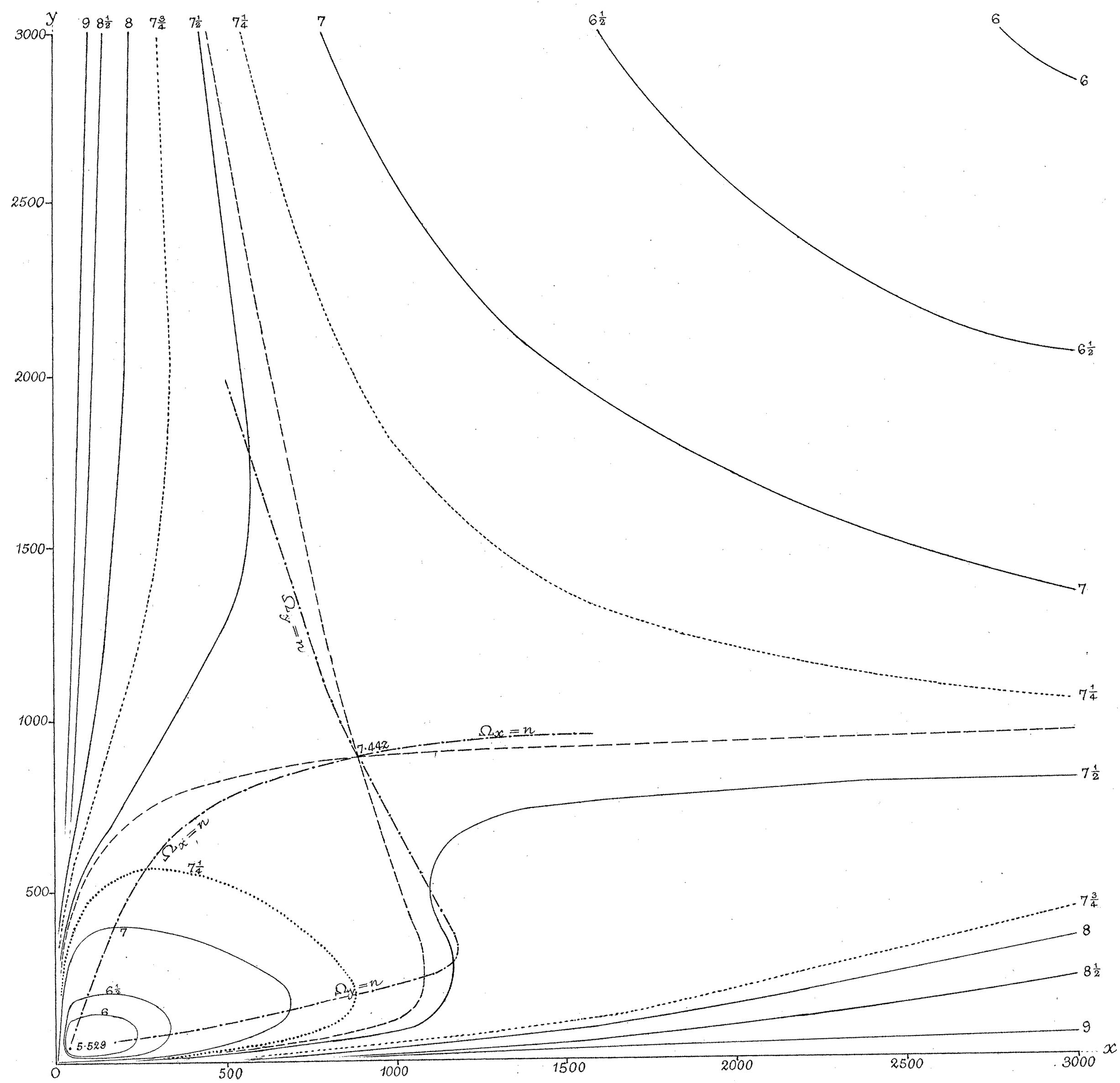
In this summary I have endeavoured not only to set forth the influence which tidal friction may, and probably has had in the history of the system, but also to point out what effects it cannot have produced.

The present investigations afford no grounds for the rejection of the nebular hypothesis, but while they present evidence in favour of the main outlines of that theory, they introduce modifications of considerable importance.

Tidal friction is a cause of change of which LAPLACE's theory took no account,* and although the activity of that cause is to be regarded as mainly belonging to a later period than the events described in the nebular hypothesis, yet its influence has been of great, and in one instance of even paramount importance in determining the present condition of the planets and their satellites.

* Note added on July 28, 1881.

Dr. T. R. MAYER appears to have been amongst the first, if not quite the first, to draw attention to the effects of tidal friction. I have recently had my attention called to his paper on "Celestial Dynamics" [Translation, 'Phil. Mag.,' 1863, vol. 25, pp. 241, 387, 417], in which he has preceded me in some of the remarks made above. He points out that, as the joint result of contraction and tidal friction, "the whole life of the earth therefore may be divided into three periods—youth with increasing, middle age with uniform, and old age with decreasing velocity of rotation."

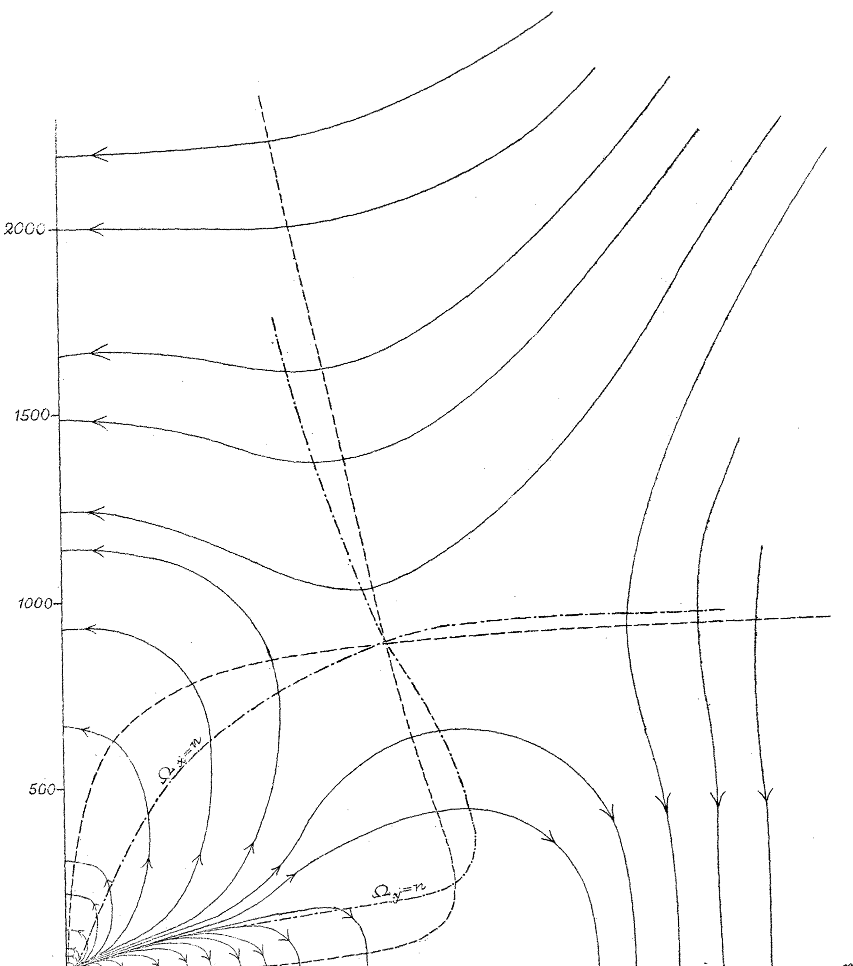


Contours of the surface $2z = (9 - x^{\frac{1}{7}} - 2y^{\frac{1}{7}})^2 - 20\left(\frac{1}{x^{\frac{2}{7}}} + \frac{2}{y^{\frac{2}{7}}}\right)$ when x & y are both positive.

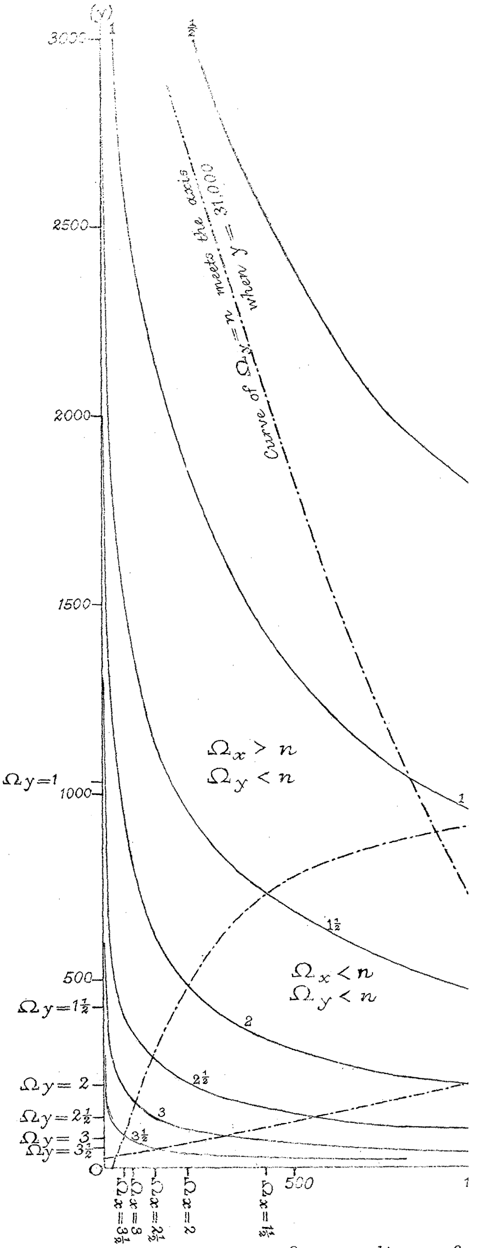
N.B. The values of $2z$ indicated by the numbers on the Contour-lines are all negative, so that the smaller numbers indicate higher Contours.

West, Newman & Co. lith.

Fig. 1.



Lines of greatest slope on the surface $2z = (9 - x^{\frac{1}{7}} - 2y^{\frac{1}{7}})^2 - 20\left(\frac{1}{x^{\frac{2}{7}}} + \frac{2}{y^{\frac{2}{7}}}\right)$

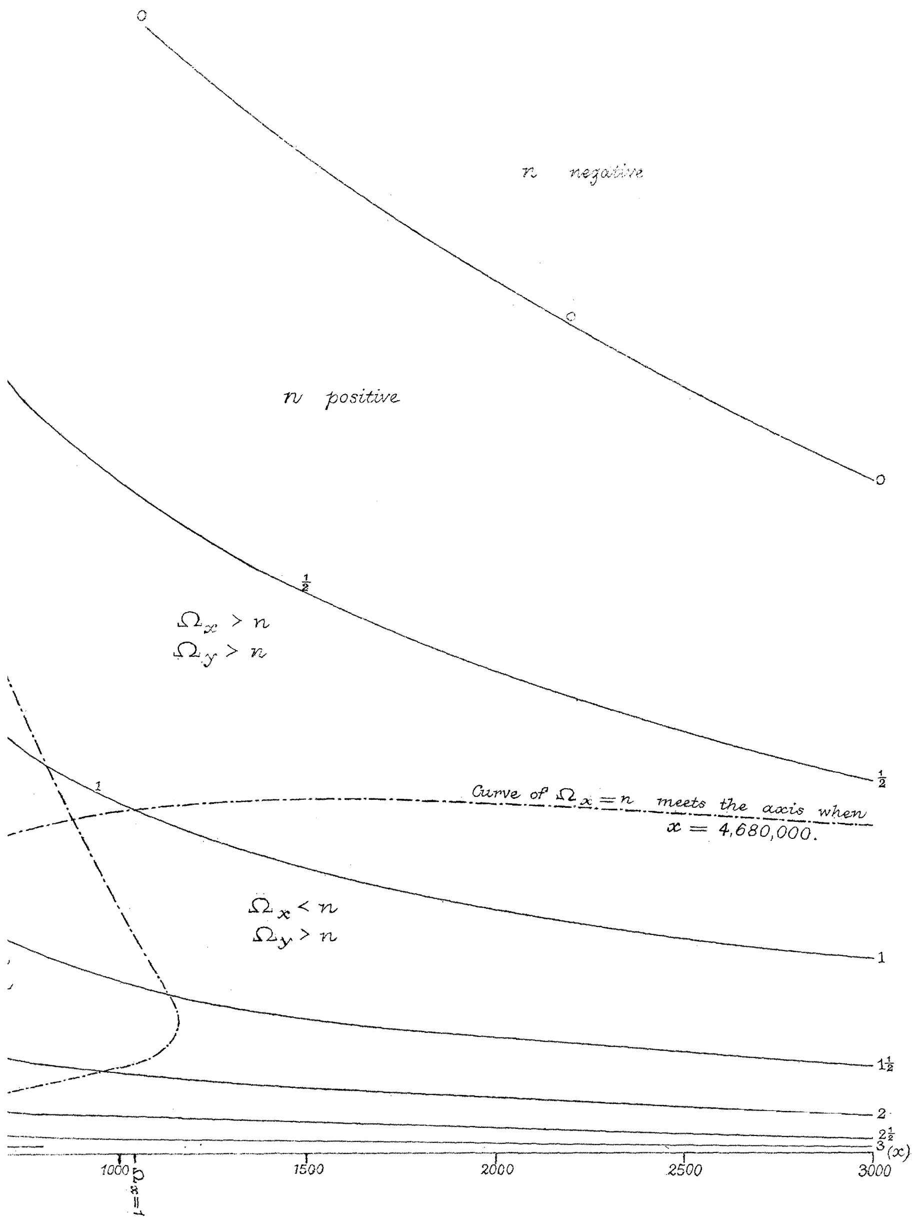


Contour lines of

West, Newman & Co. lith.

G. Darwin.

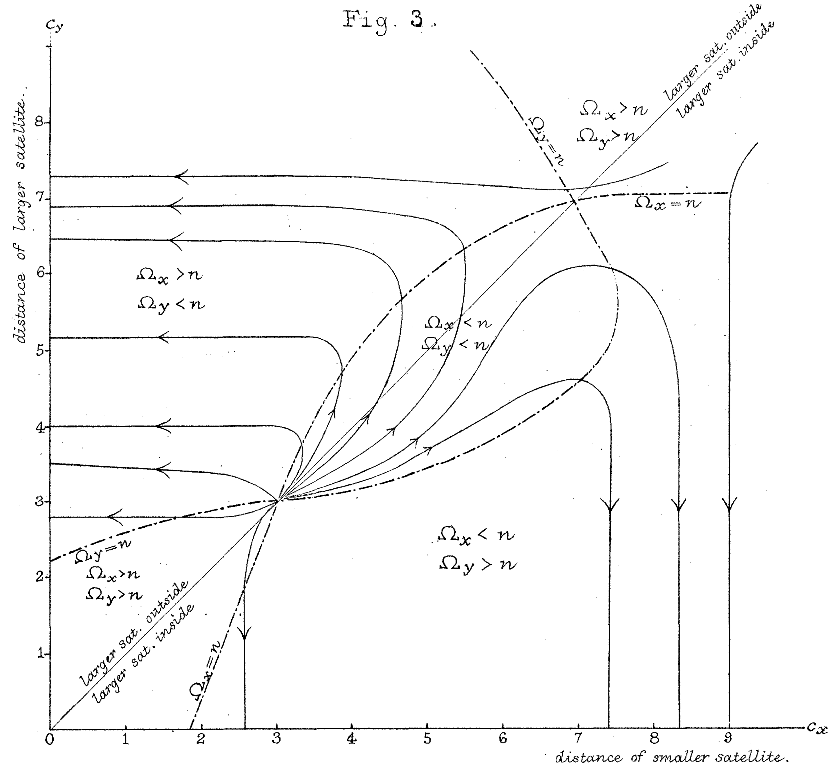
Fig. 2.



lines of the surface $n = 9 - x^{\frac{1}{7}} - 2y^{\frac{1}{7}}$

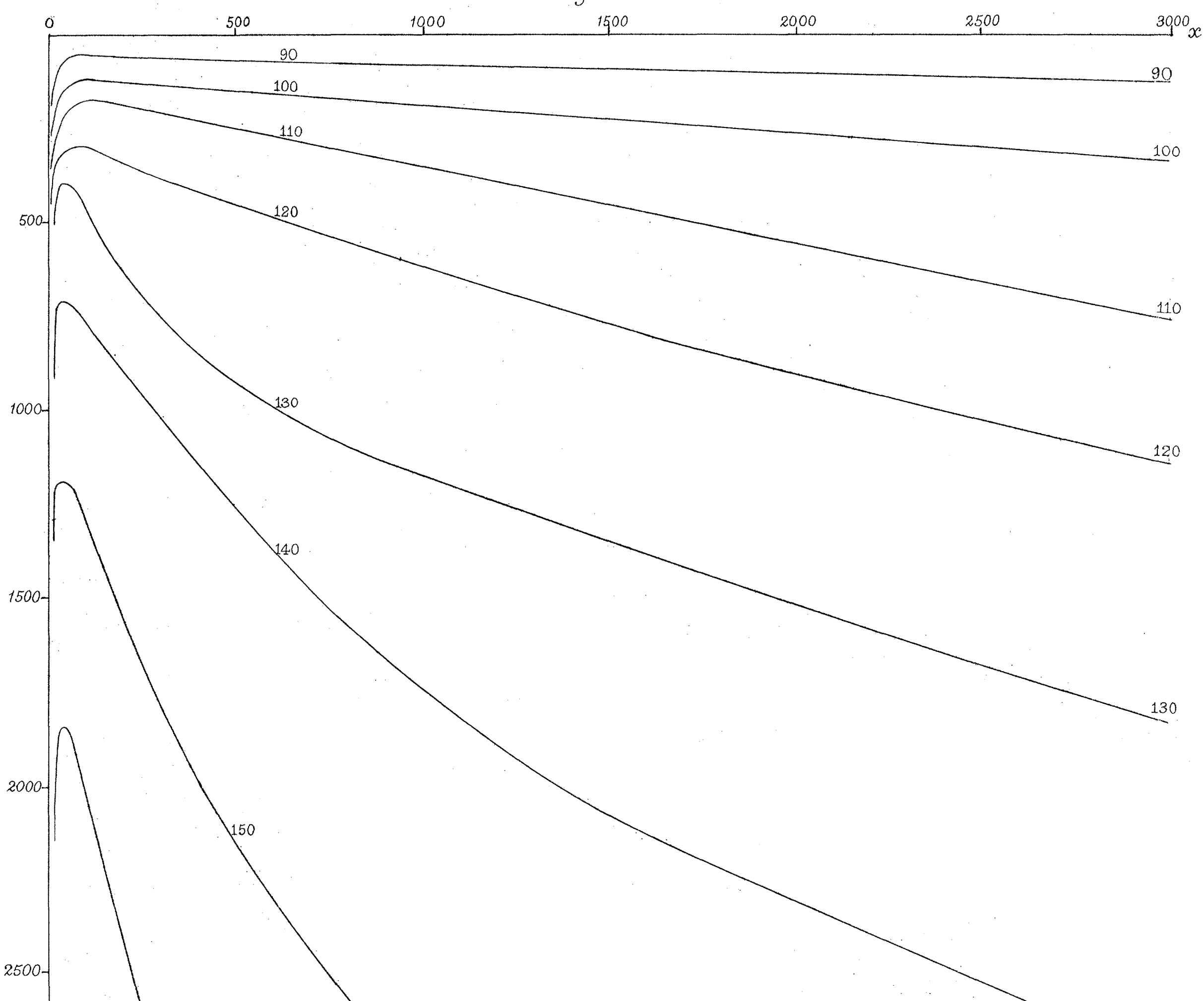
Phil. Trans. 1881. Plate 62.

Fig. 3.



c_y
distance of larger satellite.
c_x
distance of smaller satellite.
0
1
2
3
4
5
6
7
8
9
$\Omega_x > n$
$\Omega_y > n$
$\Omega_y = n$
$\Omega_x = n$
$\Omega_x > n$
$\Omega_y < n$
$\Omega_x < n$
$\Omega_y < n$
$\Omega_x < n$
$\Omega_y > n$
$\Omega_y = n$
$\Omega_x > n$
$\Omega_y > n$
Larger sat. outside
Larger sat. inside

Fig. 4.



0
500
1000
1500
2000
2500
3000
x
90
100
110
120
130
140
150

1881. *Plate* 63.

3000 *x*

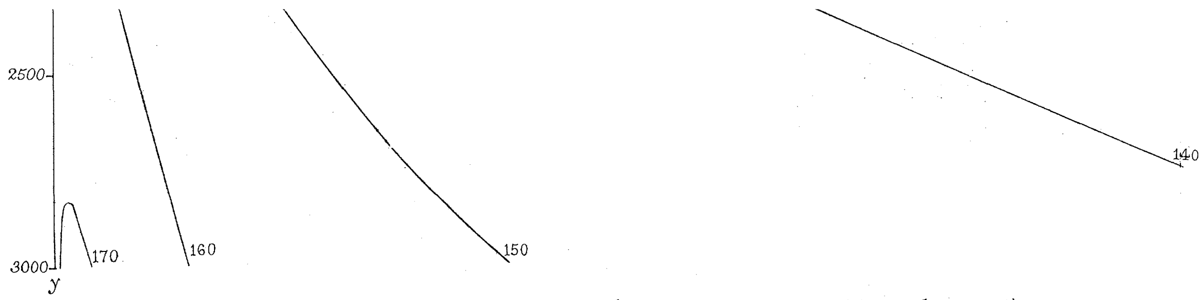
90

100

110

120

130



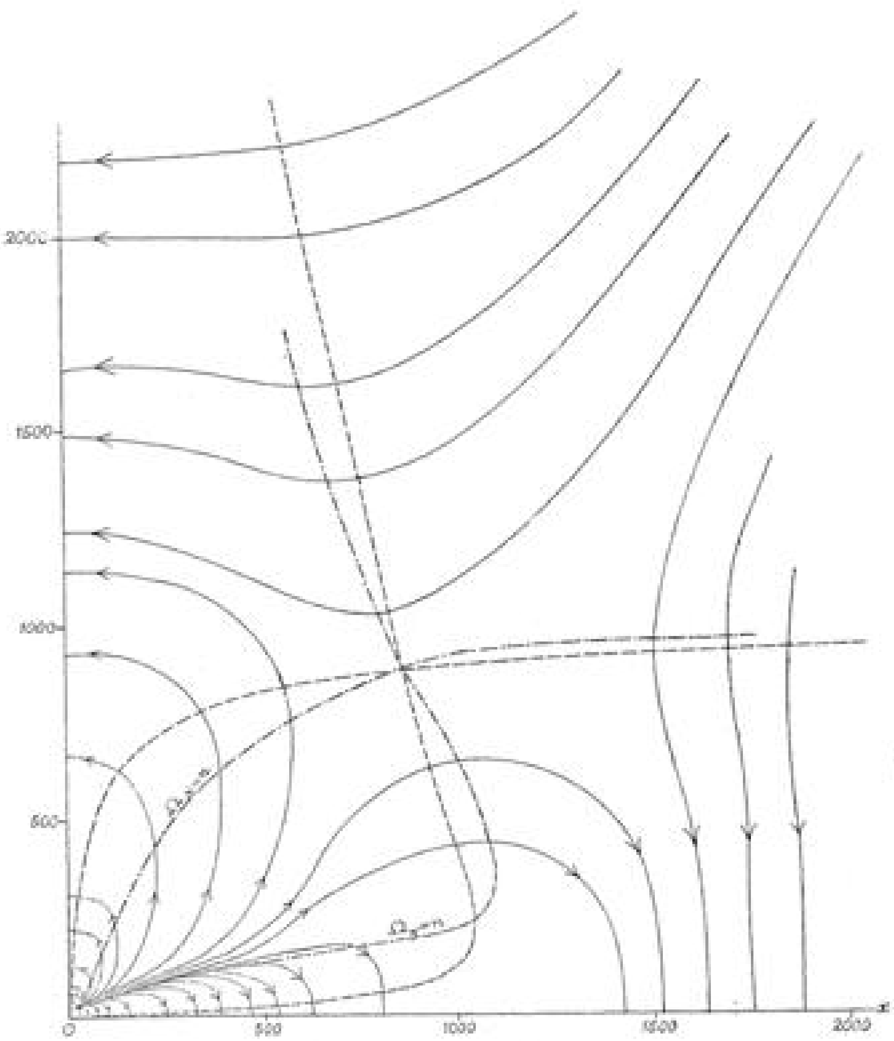
Contour lines of the surface $2z = (9 - x^{\frac{1}{7}} - 2y^{\frac{1}{7}})^2 - 20\left(\frac{1}{x^{\frac{2}{7}}} + \frac{2}{y^{\frac{2}{7}}}\right)$ *when x is positive and y negative.*

West, N

140

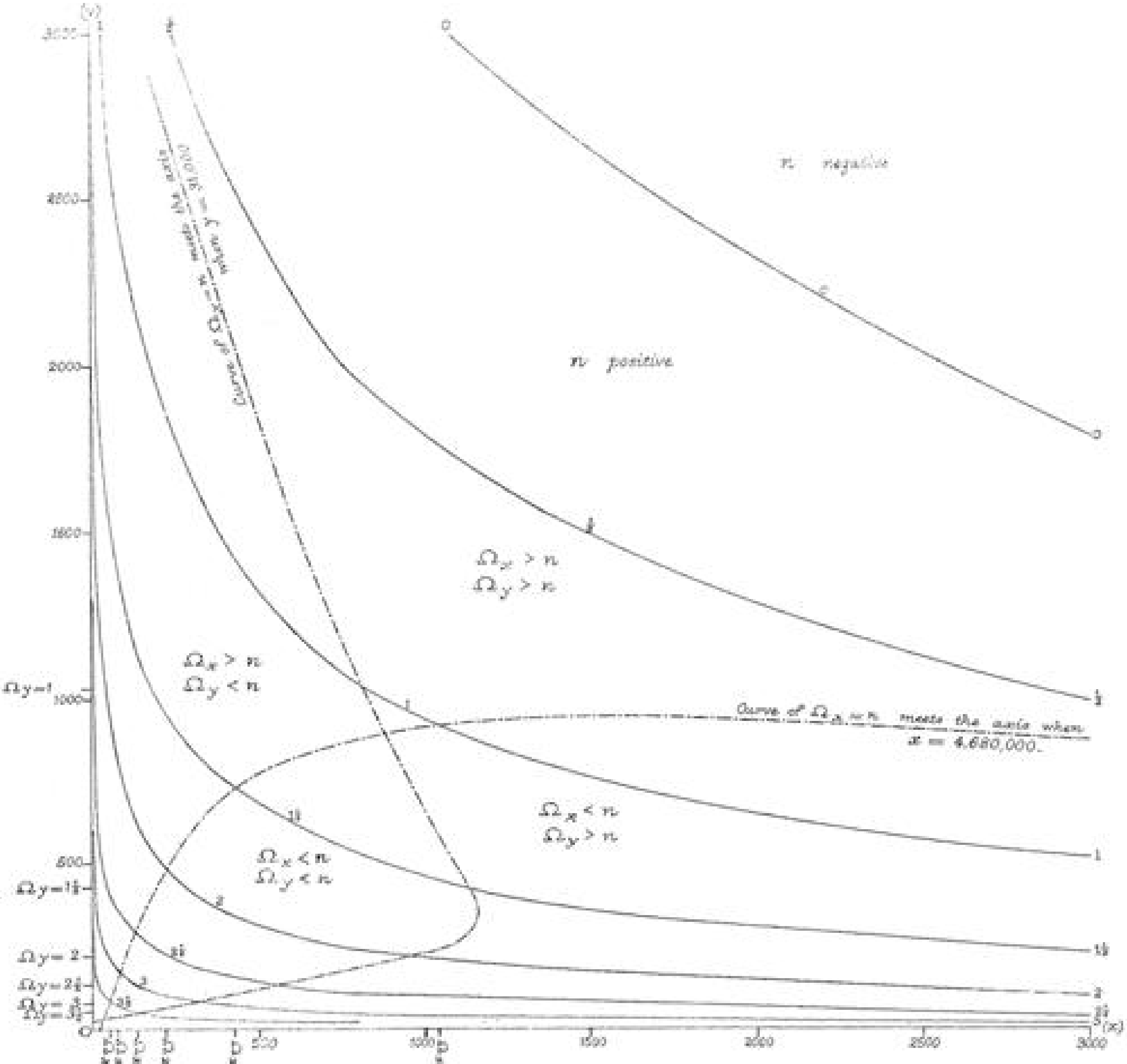
West, Newman & C° lith.

Fig. 1.



Lines of greatest slope on the surface $2z = (9 - x^{\frac{1}{3}} - 2y^{\frac{1}{3}})^2 - 20\left(\frac{1}{x^{\frac{1}{3}}} + \frac{2}{y^{\frac{1}{3}}}\right)$

Fig. 2.



Contour lines of the surface $n = 9 - x^{\frac{1}{3}} - 2y^{\frac{1}{3}}$

Fig. 3.

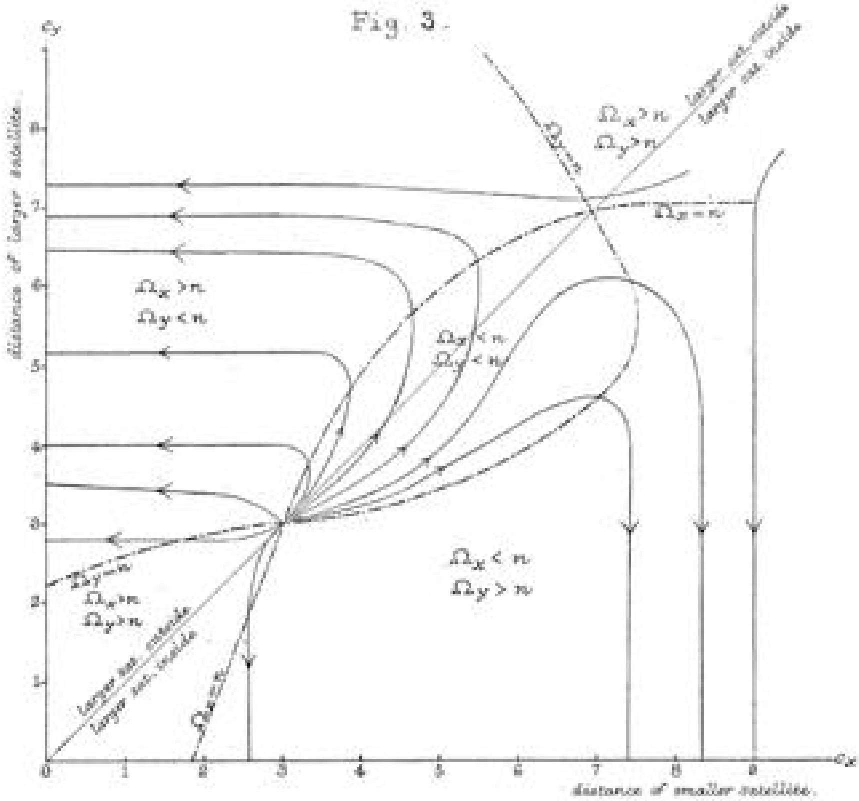
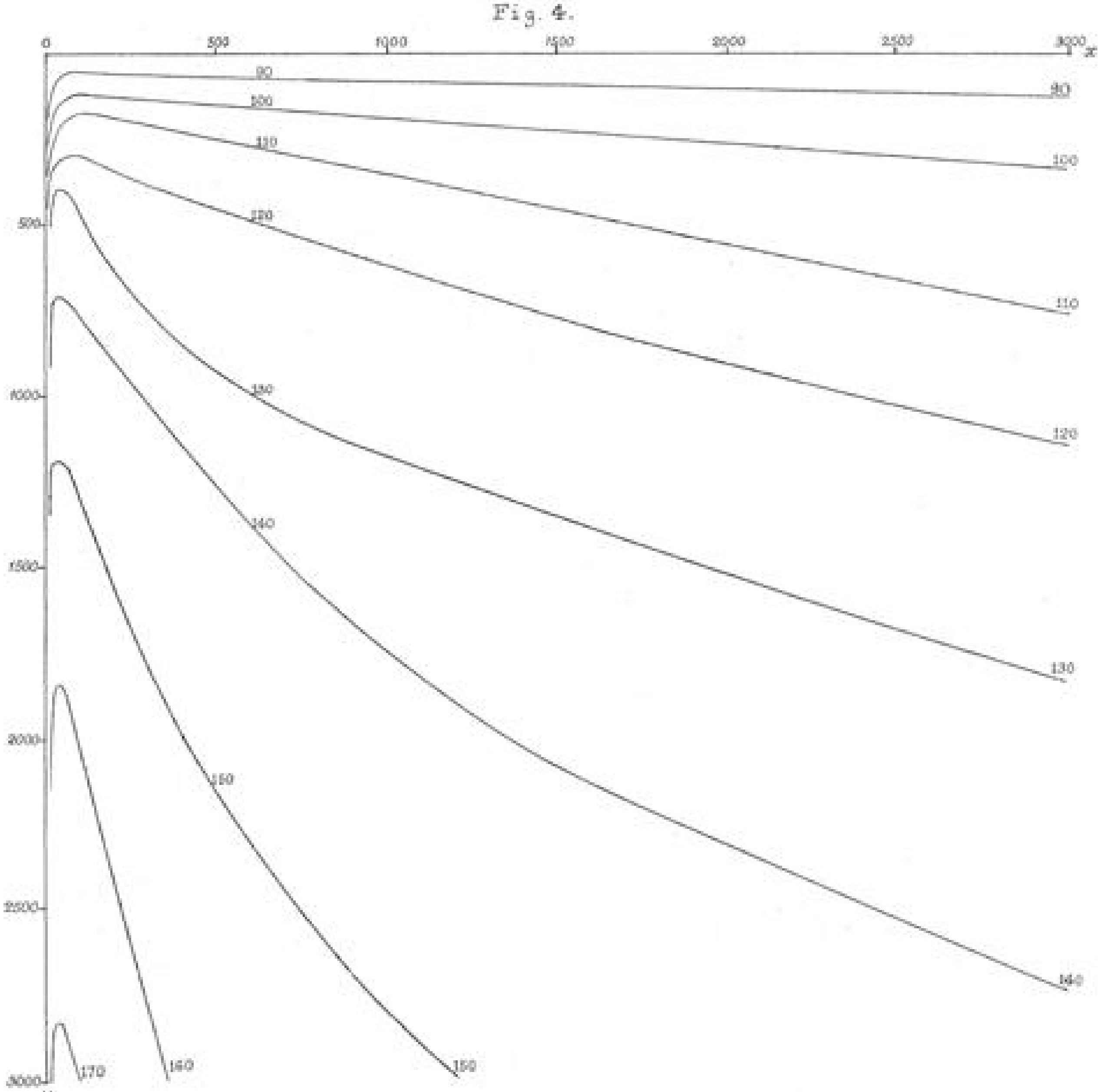


Fig. 4.



Contour lines of the surface $2z = (9 - x^{\frac{1}{2}} - 2y^{\frac{1}{2}})^2 - 20\left(\frac{1}{x^{\frac{1}{2}}} + \frac{8}{y^{\frac{1}{2}}}\right)$ *when* x *is positive and* y *negative.*

West, Newman & Co. lith.